

CONTROL SYSTEMS - Fall 2008

Problem Set 5 Solutions

1. The controllability matrix is

$$C_{AB} = \begin{bmatrix} \alpha & 0 \\ 1 & \alpha - 1 \end{bmatrix}$$

which is nonsingular since $\alpha \neq 1$. The characteristic polynomial of A is $s(s+1) = s^2 + s$. The desired characteristic polynomial is $r(s) = (s+2)(s+3) = s^2 + 5s + 6$. If the system were in controllable canonical form, the desired feedback gain $k_c^T = [6 \quad 4]$. The required matrix V which transforms (A, B) into controllable canonical form is given by

$$\begin{aligned} V &= C_{AB} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha & 0 \\ 1 & \alpha - 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} \alpha & \alpha \\ \alpha & 1 \end{bmatrix} \end{aligned}$$

$$V^{-1} = \frac{1}{\alpha(1-\alpha)} \begin{bmatrix} 1 & -\alpha \\ -\alpha & \alpha \end{bmatrix}$$

Finally, the desired feedback gain k^T is given by

$$\begin{aligned} k^T &= k_c^T V^{-1} \\ &= [6 \quad 4] \frac{1}{\alpha(1-\alpha)} \begin{bmatrix} 1 & -\alpha \\ -\alpha & \alpha \end{bmatrix} \\ &= \begin{bmatrix} \frac{6-4\alpha}{\alpha(1-\alpha)} & \frac{-2}{1-\alpha} \end{bmatrix} \end{aligned}$$

The closed loop system matrix $A - Bk^T$ is given by

$$A - Bk^T = \begin{bmatrix} -\frac{6-4\alpha}{1-\alpha} & \frac{2\alpha}{1-\alpha} \\ -\frac{\alpha^2-5\alpha+6}{\alpha(1-\alpha)} & \frac{1+\alpha}{1-\alpha} \end{bmatrix}$$

One can verify, after some algebra, that $\det(sI - A + Bk^T) = s^2 + 5s + 6$, as desired.

2.

$$C_{AB} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

Hence (A, B) is controllable.

$$\begin{aligned} \det(sI - A) &= \det \begin{bmatrix} s & -1 & -1 \\ -1 & s+1 & 0 \\ 0 & -2 & s \end{bmatrix} \\ &= s(s^2 + s - 1) - 2 = s^3 + s^2 - s - 2 \end{aligned}$$

The transformation which brings (A, B) to controllable canonical form is

$$\begin{aligned}
 V &= [A^2B \ AB \ B] \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
 V^{-1} &= \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}
 \end{aligned}$$

The desired char. poly. $r(s) = (s + 1)^3 = s^3 + 3s^2 + 3s + 1$. Thus

$$\begin{aligned}
 K &= [3 \ 4 \ 2]V^{-1} \\
 &= [3 \ 4 \ 2] \begin{bmatrix} -1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix} = [1 \ 1 \ 1] \\
 A - BK &= \begin{bmatrix} 0 & 1 & 1 \\ 1 & -1 & 0 \\ 0 & 2 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & -1 \end{bmatrix}
 \end{aligned}$$

3. We have the following differential equations

$$\begin{aligned}
 \dot{x}_1 &= x_2 \\
 \dot{x}_2 + 2x_2 &= \dot{x}_3 + x_3 \\
 2\dot{x}_3 + x_3 &= u
 \end{aligned}$$

$$\begin{aligned}
 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
 \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \\
 &= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & -2 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix} u
 \end{aligned}$$

$$\begin{aligned}\det(sI - A) &= \det \begin{bmatrix} s & -1 & 0 \\ 0 & s+2 & -\frac{1}{2} \\ 0 & 0 & s+\frac{1}{2} \end{bmatrix} = s(s+2)(s+\frac{1}{2}) \\ &= s^3 + \frac{5}{2}s^2 + s\end{aligned}$$

$$\begin{aligned}V &= [A^2B \quad AB \quad B] \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 1 & 0 \\ 1 & \frac{5}{2} & 1 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{3}{4} & \frac{1}{2} & 0 \\ \frac{11}{8} & -\frac{3}{4} & \frac{1}{2} \\ \frac{1}{8} & -\frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 1 & 0 \\ 1 & \frac{5}{2} & 1 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} -6 & 4 & 0 \\ 11 & -6 & 4 \\ 1 & -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \frac{5}{2} & 1 & 0 \\ 1 & \frac{5}{2} & 1 \end{bmatrix} \\ &= \frac{1}{8} \begin{bmatrix} 4 & 4 & 0 \\ 0 & 4 & 4 \\ 0 & 8 & 4 \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 1 & 0.5 \end{bmatrix} \\ \therefore V^{-1} &= \begin{bmatrix} 2 & 2 & -2 \\ 0 & -2 & 2 \\ 0 & 4 & -2 \end{bmatrix}\end{aligned}$$

Desired $r(s) = (s+1)[(s+3)^2 + 1] = (s+1)(s^2 + 6s + 10) = s^3 + 7s^2 + 16s + 10$

$$\begin{aligned}k &= [r_3 - p_3 \quad r_2 - p_2 \quad r_1 - p_1]V^{-1} \\ &= [10 \quad 15 \quad \frac{9}{2}] \begin{bmatrix} 2 & 2 & -2 \\ 0 & -2 & 2 \\ 0 & 4 & -2 \end{bmatrix} = [20 \quad 8 \quad 1]\end{aligned}$$

4. $A = \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ -4 & 0 & -1 \end{bmatrix} \quad B = \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & -2 \end{bmatrix}$

$$[b_1 \quad Ab_1 \quad A^2b_1] = \begin{bmatrix} -2 & -4 & -6 \\ 2 & 4 & 6 \\ 2 & 6 & 10 \end{bmatrix} \quad \text{which has rank 2}$$

$$[b_1 \quad Ab_2 \quad A^2b_2] = \begin{bmatrix} 2 & 6 & \star \\ -1 & -3 & \star \\ -2 & -6 & \star \end{bmatrix} \quad \text{which has rank 1}$$

$$Q = \begin{bmatrix} -2 & -4 & 2 \\ 2 & 4 & -1 \\ 2 & 6 & -2 \end{bmatrix} \quad Q^{-1} = \begin{bmatrix} -\frac{1}{2} & 1 & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 1 & 0 \end{bmatrix}$$

We write down

$$\begin{aligned}
 S &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} & K_1 &= -SQ^{-1} = \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \\
 A - BK_1 &= \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ -4 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -2 & 2 \\ 2 & -1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} \\
 &= \begin{bmatrix} 5 & 2 & 1 \\ -2 & 1 & -1 \\ -4 & 0 & -1 \end{bmatrix} - \begin{bmatrix} -1 & 0 & -1 \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 6 & 2 & 2 \\ -\frac{5}{2} & 1 & -\frac{3}{2} \\ -5 & 0 & -2 \end{bmatrix}
 \end{aligned}$$

$$\det(sI - A + BK_1) = s^3 - 5s^2 + 7s - 3$$

$$\text{Desired } r(s) = (s + 1)((s + 2)^2 + 1) = s^3 + 5s^2 + 9s + 5$$

$$V = \begin{bmatrix} -4 & -4 & -2 \\ 5 & 4 & 2 \\ 8 & 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 7 & -5 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 6 & -2 \\ -1 & -6 & 2 \\ -8 & -4 & 2 \end{bmatrix}$$

$$V^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 3.5 & 3 & 0.5 \\ 11 & 10 & 1.5 \end{bmatrix}$$

$$k_1 = [8 \ 2 \ 10]V^{-1} = [125 \ 114 \ 16]$$

$$K = K_1 + e_1 k_1 = \begin{bmatrix} 125 & 114 & 16 \\ -\frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

One can verify that the closed loop system matrix

$$A - BK = \begin{bmatrix} 256 & 230 & 34 \\ -252.5 & -227 & -33.5 \\ -255 & -228 & -34 \end{bmatrix}$$

does have eigenvalues at $-1, -2 \pm i$.

5. (a) The Q matrix is readily found to be

$$Q = [b_1 \ b_2 \ Ab_2] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -2 \\ 1 & 0 & 0 \end{bmatrix}$$

Hence

$$S = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

The structure of Q allows its inverse to be easily found:

$$Q^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

This gives

$$K_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

resulting in

$$A_1 = A - BK_1 = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 4 \end{bmatrix}$$

Straightforward calculations give

$$C_{A_1 b_1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 16 \end{bmatrix}$$
$$\det(sI - A_1) = s^4 - 12s^2 - 16$$

giving

$$V = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} -12 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 1 \end{bmatrix}$$

The desired characteristic polynomial is $r(s) = (s+2)(s+3)(s+4) = s^3 + 9s^2 + 26s + 24$. Hence

$$k_c^T = [24 - (-16) \quad 26 - (-12) \quad 9] = [40 \quad 38 \quad 9]$$
$$k_1^T = k_c^T V^{-1} = [40 \quad 38 \quad 9] \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -4 & 1 \end{bmatrix} = [0 \quad 2 \quad 9]$$

Finally,

$$K = \begin{bmatrix} 0 & 2 & 9 \\ 0 & 0 & -1 \end{bmatrix}$$

resulting in

$$A - BK = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & -5 \end{bmatrix}$$

with eigenvalues at $-2, -3, -4$.

- (b) Note that the structure of (A, B) shows that the eigenvalue 4 can be controlled from the 1st column of B , while the eigenvalues $-2, -2$ can be controlled from the 2nd column of B . In particular, we can use the first input component u_1 to shift the eigenvalue 4 to -4 . and we need only to use u_2 to shift the -2 entry in the $(2, 2)$ position of A to -3 , and leave the -2 in the $(1, 1)$ position unchanged. Combining, we see that the required feedback K is given by

$$K = \begin{bmatrix} 0 & 0 & 8 \\ 0 & 1 & 0 \end{bmatrix}$$

This yields

$$A - BK = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -4 \end{bmatrix}$$

with eigenvalues $-2, -3, -4$, as desired! Another possible choice is

$$K = \begin{bmatrix} 0 & 0 & 7 \\ 0 & 2 & 0 \end{bmatrix}$$

This yields

$$A - BK = \begin{bmatrix} -2 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

(c) If you use the Matlab command “place”, the resulting feedback K is

$$K = \begin{bmatrix} 0 & 0 & 6 \\ 2 & 3 & 0 \end{bmatrix}$$

resulting in

$$A - BK = \begin{bmatrix} -2 & 1 & 0 \\ -2 & -5 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

with eigenvalues at $-2, -3, -4$. In effect, this corresponds to using u_1 to shift the pole at 4 to -2 and using u_2 to shift the poles at $-2, -2$ to -3 and -4 .

(d) The norms of the various K 's are as follows:

$$\left\| \begin{bmatrix} 0 & 2 & 9 \\ 0 & 0 & -1 \end{bmatrix} \right\| = 9.2711$$

$$\left\| \begin{bmatrix} 0 & 0 & 8 \\ 0 & 1 & 0 \end{bmatrix} \right\| = 8$$

$$\left\| \begin{bmatrix} 0 & 0 & 7 \\ 0 & 2 & 0 \end{bmatrix} \right\| = 7$$

$$\left\| \begin{bmatrix} 0 & 0 & 6 \\ 2 & 3 & 0 \end{bmatrix} \right\| = 6$$

The K from the multivariable algorithm of Chapter3 has the largest norm. In a sense, it is using the control inputs least efficiently.

$$6. \quad (i) \quad A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$C_{AB} = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & \dots \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is of rank 2, so that the system is not controllable.

(ii) To check stabilizability using the PBH test, first determine the eigenvalues of A .

$$\begin{aligned} \det(sI - A) &= \begin{vmatrix} s & -1 & 0 & -1 \\ 0 & s & -1 & 0 \\ 0 & -2 & s+1 & 0 \\ 0 & 1 & 1 & s-1 \end{vmatrix} \\ &= s(s-1)(s^2 + s - 2) = s(s-1)^2(s+2) \end{aligned}$$

There are 2 unstable eigenvalues 0 and 1. Choose the eigenvalue 1 to check stabilizability. Form

$$[A - I \quad B] = \begin{bmatrix} -1 & 1 & 0 & 1 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 2 & -2 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

It is clear that $[A - I \ B]$ has 3 linearly independent columns, namely,

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

We conclude from the PBH test that the system is not stabilizable.