CONTROL SYSTEMS - Fall 2008
Problem Set 4 Solutions

1. $C_{AB} = \begin{bmatrix} 0 & 1 & -2 \\ 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

   The first and second columns are independent, but the 3rd$=-2\times$ 2nd-1st. So the system is not controllable. The reachable set $\mathcal{R}_0 = \mathcal{R}(C_{AB})$, according to Theorem 2.1. So

   $$\mathcal{R}_0 = \mathcal{R}(C_{AB}) = span\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}\} = span\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$$

   Finding a vector x not in $\mathcal{R}_0$ is the same of finding a vector not in $span\{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\}$. Clearly, $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is one such vector.

2. $C_{AB} = \begin{bmatrix} 1 & 1 & 4 & 0 & \\ 0 & 0 & 1 & 0 & \dots \\ 0 & 1 & 1 & 1 & \\ 1 & 0 & 1 & 1 & \end{bmatrix}$

   The 1st 4 columns of C_{AB} are linearly independent so that (A, B) is controllable.

3. Compute the controllability matrix C_{AB} for the coupled 2-cart system with 1 input.

   $$C_{AB} = \begin{bmatrix} 0 & -\frac{1}{M_1} & 0 & \frac{K}{M_1^2} + \frac{K}{M_1 M_2} \\ -\frac{1}{M_1} & 0 & \frac{K}{M_1^2} + \frac{K}{M_1 M_2} & 0 \\ 0 & \frac{1}{M_2} & 0 & -\frac{K}{M_1 M_2} - \frac{K}{M_2^2} \\ \frac{1}{M_2} & 0 & -\frac{K}{M_1 M_2} - \frac{K}{M_2^2} & 0 \end{bmatrix}$$
   $$\quad B \qquad AB \qquad A^2 B \qquad A^3 B$$

   Note that for this C_{AB}

   $$\text{the 3rd column} = \begin{bmatrix} 0 \\ \frac{1}{M_1}\left(\frac{K}{M_1} + \frac{K}{M_2}\right) \\ 0 \\ -\frac{1}{M_2}\left(\frac{K}{M_1} + \frac{K}{M_2}\right) \end{bmatrix} = -\left(\frac{K}{M_1} + \frac{K}{M_2}\right)\begin{bmatrix} \text{1st} \\ \text{column} \end{bmatrix}$$

   A similar relationship holds between the second and fourth columns, so that only two columns in C_{AB} are linearly independent. Hence the system is not controllable. In the case where $M_1 = K = 1$, $M_2 = \frac{1}{2}$, we have

   $$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 0 & -2 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ -1 \\ 0 \\ 2 \end{bmatrix}$$

The controllability matrix is given by

$$C_{AB} = \begin{bmatrix} 0 & -1 & 0 & 3 \\ -1 & 0 & 3 & 0 \\ 0 & 2 & 0 & -6 \\ 2 & 0 & -6 & 0 \end{bmatrix}$$

The first 2 columns of C_{AB} are linearly independent so that

$$\mathcal{R}_0 = span\{ \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} \}$$

4. (i)

$$A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Hence

$$e^{A\tau} = \begin{bmatrix} e^{-\tau} & 0 \\ 0 & e^{\tau} \end{bmatrix}$$

The controllability Gramian W_t is given by

$$\begin{aligned} W_t &= \int_0^t e^{A\tau} B B^T e^{A^T\tau} d\tau \\ &= \int_0^t \begin{bmatrix} 0 & 0 \\ 0 & e^{2\tau} \end{bmatrix} d\tau \\ &= \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}(e^{2t} - 1) \end{bmatrix} \end{aligned}$$

For any $x = \begin{bmatrix} 0 & \alpha \end{bmatrix}^T$, the solution of the equation $W_t \xi = x$ is easily determined:

$$W_t \xi = \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{2}(e^{2t} - 1) \end{bmatrix} \xi = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}$$

giving the solution

$$\xi = \begin{bmatrix} \rho \\ \frac{2\alpha}{e^{2t}-1} \end{bmatrix}$$

where ρ is arbitrary.

(ii) For $t = 1$, $\alpha = 2$, the control input achieving the transfer is given by

$$\begin{aligned} u(\tau) &= B^T e^{A^T\tau} \xi \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-(1-\tau)} & 0 \\ 0 & e^{(1-\tau)} \end{bmatrix} \begin{bmatrix} \rho \\ \frac{4}{e^2-1} \end{bmatrix} \\ &= \frac{4e^{1-\tau}}{e^2 - 1} \end{aligned}$$

which is independent of ρ. We directly verify that this control input achieves the transfer at $t = 1$.

$$
\begin{aligned}
x(1) &= \int_0^1 \begin{bmatrix} e^{-(1-\tau)} & 0 \\ 0 & e^{(1-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \frac{4e^{(1-\tau)}}{e^2-1} d\tau \\
&= \int_0^1 \begin{bmatrix} 0 \\ \frac{4}{e^2-1} e^{2(1-\tau)} \end{bmatrix} d\tau \\
&= \int_0^1 \begin{bmatrix} 0 \\ \frac{4}{e^2-1} e^{2\tau} \end{bmatrix} d\tau \\
&= \begin{bmatrix} 0 \\ 2 \end{bmatrix}
\end{aligned}
$$

Remark: By Theorem 2.1, the set of reachable states consists of all those x for which the equation $C_{AB}\zeta = x$ has a solution for ζ. In the language of linear algebra, this is equivalent to saying x is reachable if and only if x is in the range space of C_{AB}. For the example,

$$
C_{AB} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}
$$

so that the equation

$$
\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \zeta = \begin{bmatrix} 0 \\ \alpha \end{bmatrix}
$$

indeed has a solution for ζ for any α, so that $\begin{bmatrix} 0 & \alpha \end{bmatrix}^T$ is reachable. Notice that solving $C_{AB}\zeta = x$ would not give you a direct way of finding the control input that achieves the transfer. However, Theorem 2.1 also shows that the equation $C_{AB}\zeta = x$ has a solution for ζ if and only if $W_t\xi = x$ has a solution for ξ. It is this solution ξ which features in the formula for determining the input that achieves the transfer.