# CONTROL SYSTEMS - Fall 2008

## Problem Set 3 Solutions

1.

$$\begin{aligned} \det(sI - A) &= \det \begin{bmatrix} s+2 & 2 & 0 \\ 0 & s & -1 \\ 0 & 3 & s+4 \end{bmatrix}^{-1} \\ &= (s+2)(s^2+4s+3) = (s+1)(s+2)(s+3) = s^3+6s^2+11s+6 \end{aligned}$$

Since $(sI - A)$ has 3 zeros, we can compute its inverse by cofactor expansion to obtain

$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s+2 & 2 & 0 \\ 0 & s & -1 \\ 0 & 3 & s+4 \end{bmatrix}^{-1} \\ &= \frac{1}{s^3+6s^2+11s+6} \begin{bmatrix} s^2+4s+3 & -2s-8 & -2 \\ 0 & s^2+6s+8 & s+2 \\ 0 & -3s-6 & s^2+2s \end{bmatrix} \end{aligned}$$

Since $b = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T$, $(sI - A)^{-1}b$ picks out the second column of $(sI - A)^{-1}$. The transfer function $h(s)$ from $u$ to $y$ is $c(sI - A)^{-1}b$, and $c = \begin{bmatrix} 1 & 0 & 1 \end{bmatrix}$ just adds the $(1, 2)$ entry and the $(3, 2)$ entry of $(sI - A)^{-1}$. Hence

$$h(s) = -\frac{5s+14}{s^3+6s^2+11s+6}$$

2. (a) Write $\frac{s-1}{s+1} = 1 - \frac{2}{s+1}$. We can realize the transfer function using the state space equation

$$\begin{aligned} \dot{x}_1 &= -x_1 - 2u \\ v &= x_1 + u \end{aligned}$$

Hence $a_1 = -1$, $b_1 = -2$, $c_1 = 1$, and $d_1 = 1$.

(b) For the transfer function $\frac{1}{s-1}$, we can write the state space realization equation

$$\begin{aligned} \dot{x}_2 &= x_2 + v \\ y &= x_2 \end{aligned}$$

Hence $a_2 = 1$, $b_2 = 1$, $c_2 = 1$, and $d_2 = 0$.

(c) The combined system is given by

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u = Ax + Bu \\ y &= \begin{bmatrix} 0 & 1 \end{bmatrix} x = Cx \end{aligned}$$

(d) The transfer function $H(s)$ from $u$ to $y$ is given by

$$\begin{aligned} H(s) &= C(sI - A)^{-1}B \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ -1 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \frac{\begin{bmatrix} s-1 & 0 \\ 1 & s+1 \end{bmatrix}}{s^2-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} \\ &= \frac{s-1}{s^2-1} = \frac{1}{s+1} \end{aligned}$$

(e) Take $u = 0$, which is obviously bounded. Then $x(t) = e^{At}x_0$. From (d), we have

$$\begin{aligned} e^{At} &= \mathcal{L}^{-1}(sI - A)^{-1} \\ &= \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{s^2-1} & \frac{1}{s-1} \end{bmatrix} \\ &= \mathcal{L}^{-1}\begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{0.5}{s-1} - \frac{0.5}{s+1} & \frac{1}{s-1} \end{bmatrix} \\ &= \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} & e^{t} \end{bmatrix} \end{aligned}$$

Let $x_0 = \begin{bmatrix} \xi_1 & \xi_2 \end{bmatrix}^T$. Then

$$\begin{aligned} y(t) &= Ce^{At}x_0 \\ &= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ \frac{1}{2}e^{t} - \frac{1}{2}e^{-t} & e^{t} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} \\ &= (\frac{1}{2}e^{t} - \frac{1}{2}e^{-t})\xi_1 + e^{t}\xi_2 \\ &= (\frac{1}{2}\xi_1 + \xi_2)e^{t} - \frac{1}{2}\xi_1 e^{-t} \end{aligned}$$

Thus for all initial conditions such that $\frac{1}{2}\xi_1 + \xi_2 \neq 0$, the output $y = x_2$ will grow exponentially without bound, although $x_1$ will remain bounded (Note: problem statement not correct for $x_1$ as it was pasted from a slightly different version of the problem).

3. (a) The particular solution for $u = 1$ satisfies

$$\dot{x}_p = Ax_p + Bu = 0$$

i.e.,

$$x_p = -A^{-1}Bu = -\begin{bmatrix} 0 & 2 \\ -1 & -3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -3 & -2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b)

$$x(0) = x_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \alpha + x_p = \alpha + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence

$$\alpha = -x_p = -\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(c) Combining (a) and (b), we get that the solution is given by

$$\begin{aligned} x(t) &= e^{At}\alpha + x_p \\ &= -\begin{bmatrix} 2e^{-t} - e^{-2t} & 2e^{-t} - 2e^{-2t} \\ -e^{-t} + e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 - 2e^{-t} + e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix} \end{aligned}$$

(d) For

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

we need to seek a particular solution of the form

$$x_p = \alpha t + \beta = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} t + \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$$

Then $\alpha$ and $\beta$ satisfy, for $u = 1$,

$$\dot{x}_p = \alpha = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \alpha_1 t + \beta_1 \\ \alpha_2 t + \beta_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We get the following equations

$$\begin{aligned} \alpha_1 &= \alpha_2 t + \beta_2 \\ \alpha_2 &= -2\alpha_2 t - 2\beta_2 + 1 \end{aligned}$$

From these equations, we see that $\alpha_2 = 0$, $\beta_2 = \frac{1}{2}$, and $\alpha_1 = \beta_2 = \frac{1}{2}$. $\beta_1$ is arbitrary and can be taken to be 0. Combining, we see

$$x_p(t) = \frac{1}{2} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

Now note

$$\begin{aligned} (sI - A)^{-1} &= \begin{bmatrix} s & -1 \\ 0 & s+2 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s+2 & 1 \\ 0 & s \end{bmatrix}}{s(s+2)} \\ &= \begin{bmatrix} \frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & \frac{1}{s+2} \end{bmatrix} = \begin{bmatrix} \frac{1}{s} & \frac{1}{2}(\frac{1}{s} - \frac{1}{s+2}) \\ 0 & \frac{1}{s+2} \end{bmatrix} \end{aligned}$$

Hence

$$e^{At} = \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix}$$

The solution to the initial value problem is $x(t) = e^{At}\gamma + x_p(t)$. Hence

$$x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \gamma + \frac{1}{2} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

This gives

$$\gamma = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

Finally

$$\begin{aligned} x(t) &= \begin{bmatrix} 1 & \frac{1}{2}(1 - e^{-2t}) \\ 0 & e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} t \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 + \frac{1}{4}(1 - e^{-2t}) + \frac{1}{2}t \\ \frac{1}{2}(1 + e^{-2t}) \end{bmatrix} \end{aligned}$$

4. (a)

$$
\begin{aligned}
B_2 &= AB_1 + p_1 I = A + p_1 I \\
B_3 &= AB_2 + p_2 I = A(A + p_1 I) + p_2 I = A^2 + p_1 A + p_2 I
\end{aligned}
$$

Hence

$$B(s) = Is^2 + (A + p_1 I)s + A^2 + p_1 A + p_2 I = A^2 + (s + p_1)A + (s^2 + p_1 s + p_2)I$$

This gives

$$(sI - A)^{-1} = \frac{1}{p(s)} A^2 + \frac{s + p_1}{p(s)} A + \frac{s^2 + p_1 s + p_2}{p(s)} I = \sum_{k=0}^{2} g_k(s) A^k$$

(b)

$$
\begin{aligned}
e^{At} &= \mathcal{L}^{-1}(sI - A)^{-1} \\
&= \mathcal{L}^{-1} \sum_{k=0}^{2} g_k(s) A^k
\end{aligned}
$$

The inverse Laplace transform operates only on the $g_k(s)$ functions. If we write

$$\alpha_k(t) = \mathcal{L}^{-1}[g_k(s)]$$

we obtain

$$e^{At} = \sum_{k=0}^{2} \alpha_k(t) A^k$$