# CONTROL SYSTEMS - Fall 2008
## Problem Set 2 Solutions

1. The eigenvalues are

$$\lambda_1 = 4, \ \ \lambda_2 = 1.$$

Their corresponding eigenvectors are

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \ \ v_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

Then solving the equation

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

we have

$$c_1 = \frac{8}{3}, \quad c_2 = \frac{1}{3}.$$

Thus the modal decomposition is given by

$$x(t) = \frac{8}{3}e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}e^{t} \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

2. (a) If $x_{20} = 0$, then for $x_{10} > 0$, $x_1(t) \to \infty$ as $t \to \infty$, and for for $x_{10} < 0$, $x_1(t) \to -\infty$ as $t \to \infty$. Similarly, if $x_{10} = 0$, $x_2(t) \to 0$ as $t \to \infty$ for all $x_{20} \neq 0$. The sketch of the solution trajectories are given below.

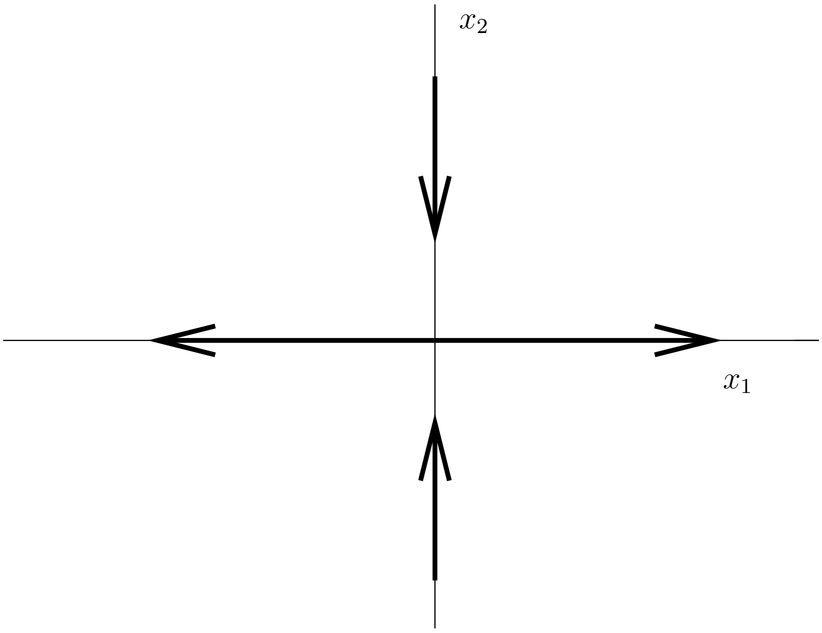


Figure 1: Solution trajectories starting on the $x_1$ or $x_2$ axes

(b) From the solutions, we can express $e^t = \frac{x_{20}}{x_2(t)}$. Hence

$$x_1(t) = x_{10}e^{2t} = x_{10}\frac{x_{20}^2}{x_2(t)^2}$$

so that

$$x_1(t)x_2(t)^2 = x_{10}x_{20}^2 = c$$

A sketch of the trajectories for $c = 0.5$ and $c = 1$ is given in the following figure.

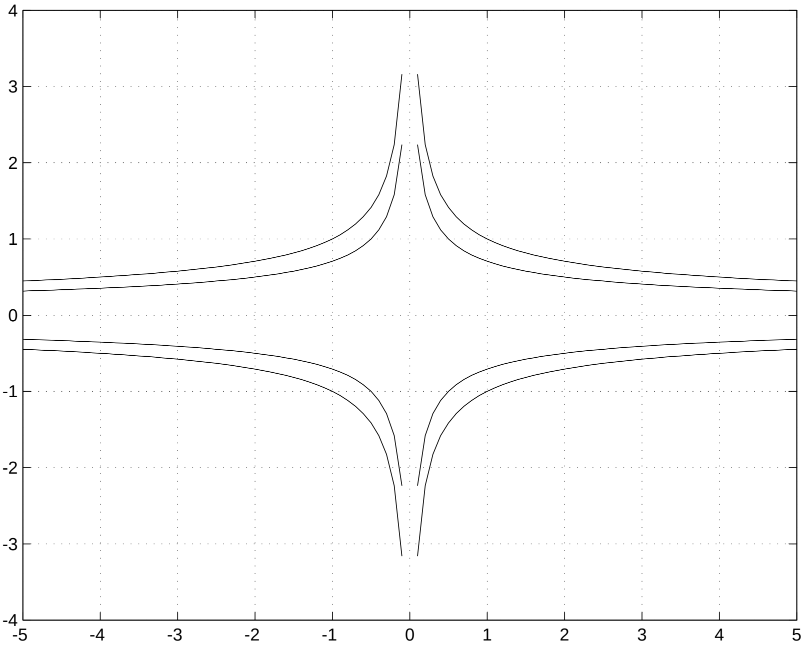


Figure 2: Solution trajectories not starting on the $x_1$ or $x_2$ axes

(c) It is straightforward to verify that

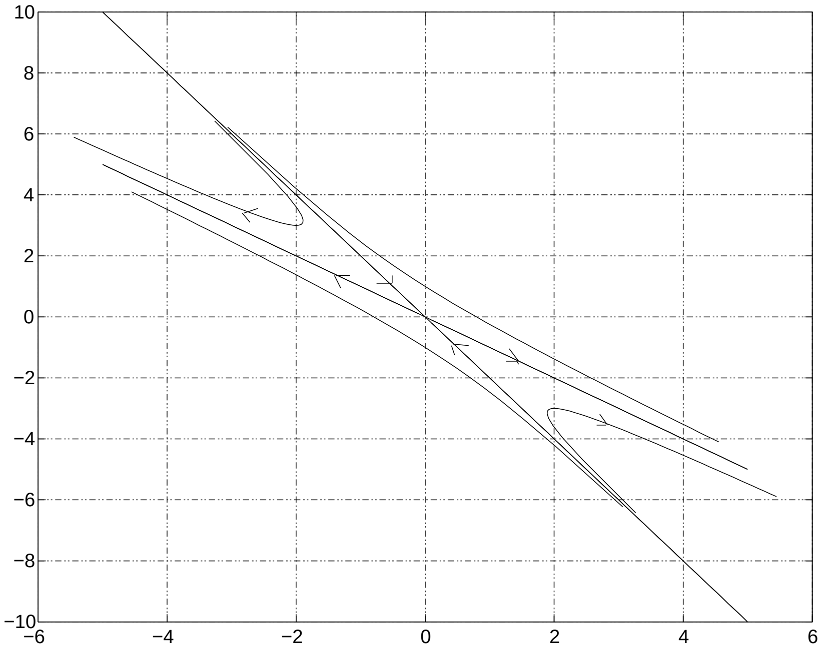
$$P^{-1} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$z$ and $x$ are related through $z = P^{-1}x$. Hence

$$z_1 = 2x_1 + x_2$$

$$z_2 = x_1 + x_2$$

If $z_1 = 0$, the $z_2$ axis corresponds to the line $x_2 = -2x_1$ in the $x_1 - x_2$ plane. Similarly, the $z_1$ axis corresponds to the line $x_2 = -x_1$. Along the line $x_2 = -2x_1$, solution trajectories converge to 0, while along the line $x_2 = -x_1$, solution trajectories diverges from 0. The rest of the solutions trajectories transform from the $z_1 - z_2$ plane to the $x_1 - x_2$ plane in a similar manner. A sketch of the trajectories for is given in the following figure.



3. We have:

$$\begin{aligned} (sI - A)(sI - A)^{-1} &= I \\ (sI - A)\frac{adj(sI - A)}{det(sI - A)} &= I \\ (sI - A)adj(sI - A) &= det(sI - A)I \end{aligned}$$

Write $adj(sI - A) = \sum_{k=1}^{n} B_k s^{n-k}$ and $det(sI - A) = s^n + \sum_{k=1}^{n} p_k s^{n-k}$ to get:

$$\begin{aligned} (sI - A)\sum_{k=1}^{n} B_k s^{n-k} &= Is^n + I\sum_{k=1}^{n} p_k s^{n-k} \\ \sum_{k=1}^{n} B_k s^{n-k+1} - \sum_{k=1}^{n} AB_k s^{n-k} &= Is^n + \sum_{k=1}^{n}(p_k I)s^{n-k} \\ \sum_{k=0}^{n-1} B_{k+1} s^{n-k} - \sum_{k=1}^{n} AB_k s^{n-k} &= Is^n + \sum_{k=1}^{n}(p_k I)s^{n-k} \\ B_1 s^n + \sum_{k=1}^{n-1}(B_{k+1} - AB_k)s^{n-k} - AB_n &= Is^n + \sum_{k=1}^{n-1}(p_k I)s^{n-k} + p_n I \end{aligned}$$

Equating powers of $s$, we will obtain:

$$\left|\begin{array}{l} B_1 = I \\ B_{k+1} = AB_k + p_k I \quad ; \quad 1 \le k \le n-1 \\ AB_n + p_n I = 0 \end{array}\right.$$

The first two equations can be solved recursively to determine $B_k,\ k = 1, \ldots, n$. The matrix $B_n$ thus obtained will satisfy the third equation by Cayley-Hamilton theorem:

$$A^n + \sum_{k=1}^{n} p_k A^{n-k} = 0$$

4.

$$A = \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$$

(i)

$$\begin{aligned} \det(sI - A) &= \begin{bmatrix} s-2 & 2 & -3 \\ -1 & s-1 & -1 \\ -1 & -3 & s+1 \end{bmatrix} \\ &= \det \begin{bmatrix} s-2 & 2 & -3 \\ -1 & s-1 & -1 \\ 0 & -s-2 & s+2 \end{bmatrix} \\ &= \det \begin{bmatrix} s-2 & -1 & -3 \\ -1 & s-2 & 0 \\ 0 & 0 & s+2 \end{bmatrix} \\ &= (s+2)(s^2 - 4s + 3) = (s+2)(s-1)(s-3) \\ &= s^3 - 2s^2 - 5s + 6 = s^3 + p_1 s^2 + p_2 s + p_3 = p(s) \end{aligned}$$

We use the results of problem 3 to determine $B(s) = B_1 s^2 + B_2 s + B_3$.

$$B_1 = I$$

$$B_2 = AB_1 + p_1 I = A - 2I = \begin{bmatrix} 0 & -2 & 3 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix}$$

$$\begin{aligned} B_3 = AB_2 + p_2 I &= \begin{bmatrix} 2 & -2 & 3 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 & -2 & 3 \\ 1 & -1 & 1 \\ 1 & 3 & -3 \end{bmatrix} - 5I \\ &= \begin{bmatrix} -4 & 7 & -5 \\ 2 & -5 & 1 \\ 2 & -8 & 4 \end{bmatrix} \end{aligned}$$

Combining, we find

$$B(s) = \begin{bmatrix} s^2 - 4 & -2s + 7 & 3s - 5 \\ s + 2 & s^2 - s - 5 & s + 1 \\ s + 2 & 3s - 8 & s^2 - 3s + 4 \end{bmatrix}$$

(ii) To determine $e^{At}$, we perform partial fractions expansion on the entries of $(sI-A)^{-1}$ and then take inverse Laplace transform.

$$\begin{aligned}(sI-A)^{-1} &= \frac{B(s)}{p(s)}\\ &= \begin{bmatrix} \frac{\frac{1}{2}}{s-3}+\frac{\frac{1}{2}}{s-1} & \frac{\frac{1}{10}}{s-3}+\frac{\frac{11}{15}}{s+2}-\frac{-\frac{5}{6}}{s-1} & \frac{\frac{2}{5}}{s-3}-\frac{\frac{11}{15}}{s+2}+\frac{\frac{1}{3}}{s-1}\\ \frac{\frac{1}{2}}{s-3}-\frac{\frac{1}{2}}{s-1} & \frac{\frac{1}{10}}{s-3}+\frac{\frac{1}{15}}{s+2}+\frac{\frac{5}{6}}{s-1} & \frac{\frac{2}{5}}{s-3}-\frac{\frac{1}{15}}{s+2}-\frac{\frac{1}{3}}{s-1}\\ \frac{\frac{1}{2}}{s-3}-\frac{\frac{1}{2}}{s-1} & \frac{\frac{1}{10}}{s-3}-\frac{\frac{14}{15}}{s+2}+\frac{\frac{5}{6}}{s-1} & \frac{\frac{2}{5}}{s-3}+\frac{\frac{14}{15}}{s+2}-\frac{\frac{1}{3}}{s-1} \end{bmatrix}\end{aligned}$$

Hence

$$e^{At} = \begin{bmatrix} \frac{1}{2}e^{3t}+\frac{1}{2}e^{t} & \frac{1}{10}e^{3t}+\frac{11}{15}e^{-2t}-\frac{5}{6}e^{t} & \frac{2}{5}e^{3t}-\frac{11}{15}e^{-2t}+\frac{1}{3}e^{t}\\ \frac{1}{2}e^{3t}-\frac{1}{2}e^{t} & \frac{1}{10}e^{3t}+\frac{1}{15}e^{-2t}+\frac{5}{6}e^{t} & \frac{2}{5}e^{3t}-\frac{1}{15}e^{-2t}-\frac{1}{3}e^{t}\\ \frac{1}{2}e^{3t}-\frac{1}{2}e^{t} & \frac{1}{10}e^{3t}-\frac{14}{15}e^{-2t}+\frac{5}{6}e^{t} & \frac{2}{5}e^{3t}+\frac{14}{15}e^{-2t}-\frac{1}{3}e^{t} \end{bmatrix}$$