CONTROL SYSTEMS - Fall 2008
Problem Set 1 Solutions

1. Let $x_1 = y_1$, $x_2 = \dot{y}_1$, $x_3 = \ddot{y}_1$, $x_4 = y_2$, $x_5 = \dot{y}_2$, $x_6 = y_3$. Then we have the following equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\begin{aligned} \dot{x}_3 &= \dddot{y}_1 = -\ddot{y}_1 - 2(\dot{y}_1 + \dot{y}_2) - 2(y_1 - y_3) + u_1 \\ &= -x_3 - 2x_2 - 2x_5 - 2x_1 + 2x_6 + u_1 \end{aligned}$$

$$\dot{x}_4 = \dot{y}_2 = x_5$$

$$\begin{aligned} \dot{x}_5 &= \ddot{y}_2 = -3(\dot{y}_2 - \dot{y}_1 + 2\dot{y}_3) - y_2 + y_1 + u_2 \\ &= -3x_5 + 3x_2 - 6(y_1 - y_3 + u_3) - x_4 + x_1 + u_2 \\ &= -3x_5 + 3x_2 - 6x_1 + 6x_6 - 6u_3 - x_4 + x_1 + u_2 \\ &= -5x_1 + 3x_2 - x_4 - 3x_5 + 6x_6 + u_2 - 6u_3 \end{aligned}$$

$$\dot{x}_6 = x_1 - x_6 + u_3$$

So the final state equation is given by

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ -2 & -2 & -1 & 0 & -2 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -5 & 3 & 0 & -1 & -3 & 6 \\ 1 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & -6 \\ 0 & 0 & 1 \end{bmatrix} u$$

2. The characteristic polynomial of A is given by $s^2 + 4s + 3$ so that the eigenvalues are at -1, -3. To find the eigenvector v for the eigenvalue -1, we solve

$$(A + I)v = \begin{bmatrix} 1 & 1 \\ -3 & -3 \end{bmatrix} v = 0$$

yielding $v = \begin{bmatrix} 1 & -1 \end{bmatrix}^T$. Similarly, for the eigenvector w corresponding to the eigenvalue -3, we solve

$$(A + 3I)w = \begin{bmatrix} 3 & 1 \\ -3 & -1 \end{bmatrix} w = 0$$

yielding $w = \begin{bmatrix} 1 & -3 \end{bmatrix}^T$. Hence the diagonalizing matrix P is given by

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}$$

and

$$P^{-1} = (-\frac{1}{2}) \begin{bmatrix} -3 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

Hence

$$
\begin{aligned}
e^{At} &= P \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} P^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} \frac{3}{2}e^{-t} & \frac{1}{2}e^{-t} \\ -\frac{1}{2}e^{-3t} & -\frac{1}{2}e^{-3t} \end{bmatrix} \\
&= \begin{bmatrix} \frac{3}{2}e^{-t} - \frac{1}{2}e^{-3t} & \frac{1}{2}e^{-t} - \frac{1}{2}e^{-3t} \\ -\frac{3}{2}e^{-t} + \frac{3}{2}e^{-3t} & -\frac{1}{2}e^{-t} + \frac{3}{2}e^{-3t} \end{bmatrix}
\end{aligned}
$$

A matrix A given by

$$
A = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & & & 1 \\ -p_n & -p_{n-1} & & \cdots & -p_2 & -p_1 \end{bmatrix}
$$

is called a matrix in companion form. If λ_1, λ_2, $\cdots$, λ_n, the eigenvalues of A, are distinct, the matrix which diagonalizes A is given by

$$
P = \begin{bmatrix} 1 & 1 & 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \cdots & \lambda_n \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \lambda_3^{n-1} & \lambda_4^{n-1} & \cdots & \lambda_n^{n-1} \end{bmatrix}
$$

The matrix P is callled a Vandermonde matrix, and can be shown to be nonsingular if the λ_i's are distinct.

3. A is given by

$$
A = \begin{bmatrix} -3 & -2 & 2 \\ 1 & 1 & -1 \\ -3 & -2 & 2 \end{bmatrix}
$$

We first find the eignevalues. Recall that the determinant of a matrix is unchanged if a multiple of a row is added to another row. The same holds for columns. So

$$\begin{aligned}\det(sI - A) &= \det\begin{bmatrix} s+3 & 2 & -2 \\ -1 & s-1 & 1 \\ 3 & 2 & s-2 \end{bmatrix} \\ &= \det\begin{bmatrix} s+3 & 2 & -2 \\ -1 & s-1 & 1 \\ -s & 0 & s \end{bmatrix} \\ &= \det\begin{bmatrix} s+1 & 2 & -2 \\ 0 & s-1 & 1 \\ 0 & 0 & s \end{bmatrix} \\ &= s(s+1)(s-1)\end{aligned}$$

This gives the eigenvalues of A to be $-1, 1, 0$. To determine the eigenvectors, we find a nontrivial solution v to the homogeneous equation

$$(\lambda I - A)v = 0$$

Solutions are readily obtained by doing elementary row operations on $\lambda I - A$.

(i) For $\lambda = 0$, elementary row operations of A gives

$$\begin{bmatrix} -3 & -2 & 2 \\ 1 & 1 & -1 \\ -3 & -2 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ -3 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence the eigenvector for the eigenvalue 0 is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

(ii) For $\lambda = 1$, $(I - A)v = 0$ yields easily the solution $v = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Hence the eigenvector for the eigenvalue 1 is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

(iii) For $\lambda = -1$, elementary row operations of $-I - A$ gives

$$\begin{bmatrix} 2 & 2 & -2 \\ -1 & -2 & 1 \\ 3 & 2 & -3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ -1 & -2 & 1 \\ 3 & -3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

Hence the eigenvector for the eigenvalue -1 is $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$.

The diagonalizing matrix P is therefore given by

$$P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

Using standard methods such as elementary row operations or cofactor expansion, we find

$$P^{-1} = \begin{bmatrix} 2 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

giving

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

Finally

$$\begin{aligned} e^{At} &= Pe^{\Lambda t}P^{-1} \\ &= \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-t} & 0 & 0 \\ 0 & e^{t} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ -1 & -1 & 1 \\ -1 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 2e^{-t} - e^{t} & e^{-t} - e^{t} & -e^{-t} - e^{t} \\ e^{t} - 1 & 2^{t} & -e^{t} + 1 \\ 2e^{-t} - e^{t} - 1 & e^{-t} - e^{t} & -e^{-t} + e^{t} + 1 \end{bmatrix} \end{aligned}$$

4. Solve $\det(\lambda I - A) = 0$, we get two eigenvalues

$$\lambda_1 = \sigma - i\omega, \lambda_2 = \sigma + i\omega.$$

The eigenvectors are given by (up to scalar multiplication)

$$v_1 = \begin{bmatrix} i \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

by solving the equations $(\lambda_i I - A)v_i = 0, \quad i = 1, 2.$

Let $T = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$. We have $T^{-1} = \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$.

Since

$$e^{T^{-1}ATt} = T^{-1}e^{At}T = e^{\Lambda t} = \begin{bmatrix} e^{(\sigma - i\omega)t} & 0 \\ 0 & e^{(\sigma + i\omega)t} \end{bmatrix},$$

we have

$$e^{At} = Te^{\Lambda t}T^{-1} = e^{\sigma t} \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}.$$

which is the same result as that given in the course notes.

5. (a) Let

$$A = \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix}$$

The eigenvalues are given by $-1 \pm 2i$. The eigenvector corresponding to $-1 + 2i$ satisfies the equation

$$\begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -1 + 2i \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

From the 2nd equation, we see that $v_1 = 2iv_2$ so that the eigenvector is given by

$$v = \begin{bmatrix} 2i \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} + i \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

(b) Let

$$P = \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}$$

Then

$$P^{-1} = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

with

$$D = P^{-1}AP = \begin{bmatrix} 0 & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \begin{bmatrix} -1 & -4 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -2 & -1 \end{bmatrix}$$

From problem 1, we see that

$$e^{Dt} = e^{-t} \begin{bmatrix} \cos 2t & \sin 2t \\ -\sin 2t & \cos 2t \end{bmatrix}$$

Hence

$$e^{At} = Pe^{Dt}P^{-1} = e^{-t} \begin{bmatrix} \cos 2t & -2\sin 2t \\ \frac{1}{2}\sin 2t & \cos 2t \end{bmatrix}$$