UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

ECE410F – Control Systems
Midterm Test – October 2007

SURNAME ____________________

GIVEN NAME ____________________

STUDENT NUMBER ____________________

DATE OF TEST ____________________

INSTRUCTIONS:

You may use one 8.5" $\times$ 11" aid sheet in preparing your answers.
Present your solutions in the blank space provided between questions; use the back of the PRECEDING page if more space is required.

TOTAL MARKS: 50

There are 4 questions. The value for each question or part of a question is shown in parentheses next to the question number.

MARKER'S REPORT	
1	**(out of 16)**
2	**(out of 16)**
3	**(out of 12)**
4	**(out of 6)**
TOTAL	**(out of 50)**

1(i) **(6 pts)** The matrix 2×2 A is known to have an eigenvalue $\lambda_1 = -1$ with corresponding eigenvector $v_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, and an eigenvalue $\lambda_2 = 1$ with corresponding eigenvector $v_2 = \begin{bmatrix} 5 \\ -2 \end{bmatrix}$. Determine e^{At}.

1(ii) **(6 pts)** Consider the linear system described by the equations

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where

$$A = \begin{bmatrix} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -3 & 1 \\ 0 & 0 & 0 & -3 \end{bmatrix} \quad B = \begin{bmatrix} 4 & 0 & 5 \\ 1 & 1 & 8 \\ 3 & 0 & 9 \\ 2 & 1 & 7 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 5 & 1 & 2 \\ 3 & 2 & 1 & 4 \end{bmatrix}$$

Let $h(t)$ denote the impulse response matrix from u to y, with $h(t) = Ce^{At}B$. Determine $h_{22}(t)$, the $(2,2)^{th}$ component of $h(t)$.

1(iii) **(4 pts)** For the system

$$\dot{x} = Ax,\ x(0) = x_0 \tag{1}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

find the set of all initial conditions x_0 such that the resulting solution x of equation (1) will converge to 0 as $t \to \infty$.

2(i) **(6 pts)** Consider the following (A, b) pair:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Use the PBH test to determine whether or not (A, b) is controllable.

2(ii) **(5 pts)** Consider the linear system $\dot{x} = Ax + bu, \ x(0) = x_0$ with

$$A = \begin{bmatrix} -\alpha & 0 \\ 2 & -\beta \end{bmatrix} \quad b = \begin{bmatrix} \gamma - \alpha \\ 2 \end{bmatrix}$$

Determine necessary and sufficient conditions for (A, b) to be controllable in terms of the parameters α, β, and γ.

2(iii) **(5 pts)** Let A be given by

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Can a column vector b be found such that (A, b) is controllable? Explain your answer.

3. Consider the single-input linear system

$$\dot{x} = Ax + bu$$

with

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}; \quad b = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

It is desired to find a feedback gain k^T such that the eigenvalues of $A - bk^T$ are placed at $-1, -2, -3$. You must follow the steps below to find k^T.

(i) **(6 pts)** Determine the nonsingular transformation V such that $V^{-1}AV, V^{-1}b$ is in controllable canonical form.

(ii) **(6 pts)** Using the results of (i), determine the desired feedback gain k^T, and determine the closed loop system matrix $A - bk^T$. (Hint: You may find it easier to solve a linear equation than to invert V).

4. Consider the multi-input linear system

$$\dot{x} = Ax + Bu \tag{2}$$

with

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 2 \\ -1 & 1 & 0 \end{bmatrix}; \quad B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Letting b_i denote the ith column of B, the system equation (2) can also be written as

$$\dot{x} = Ax + b_1 u_1 + b_2 u_2 \tag{3}$$

It is known that (A, B) is controllable, but (A, b_i) is not controllable $i = 1, 2$. It is desired to find a feedback gain K such that the eigenvalues of $A - BK$ are placed at $-1, -2, -4$. Rather than using the multivariable pole assignment algorithm, you are asked to exploit the structure of B to find K, following the steps below.

(i) **(2 pts)** Choose a feedback gain k_1^T such that with $u_1 = -k_1^T x$, the resulting system

$$\begin{aligned} \dot{x} &= (A - b_1 k_1^T)x + b_2 u_2 \\ &= \begin{bmatrix} 0 & 0 & 0 \\ p_1 & p_2 & p_3 \\ 0 & 0 & -4 \end{bmatrix} x + b_2 u_2 \end{aligned} \tag{4}$$

Identify the coefficients p_1, p_2, p_3.

(ii) **(3 pts)** Now choose a feedback gain k_2^T such that with $u_2 = -k_2^T x$, the resulting system

$$\dot{x} = (A - b_1 k_1^T - b_2 k_2^T)x \tag{5}$$

will have the eigenvalues of $A - b_1 k_1^T - b_2 k_2^T$ located at $-1, -2, -4$. For this part, you may use any method to find k_2^T. (Hint: It is probably easiest to set $k_2^T = \begin{bmatrix} \alpha & \beta & \gamma \end{bmatrix}$ and determine α and β directly.)

(iii) **(1 pt)** Using the results of (i) and (ii), find the gain K such that $A - BK$ will have eigenvalues at $-1, -2, -4$.