UNIVERSITY OF TORONTO
FACULTY OF APPLIED SCIENCE AND ENGINEERING
DEPARTMENT OF ELECTRICAL AND COMPUTER ENGINEERING

ECE410F – Control Systems
Midterm Test – November 2006

SURNAME ______________________________

GIVEN NAME ______________________________

STUDENT NUMBER ______________________________

DATE OF TEST ______________________________

INSTRUCTIONS:

You may use one 8.5"×11" aid sheet in preparing your answers.
You may use, if necessary, the following formula:

$$\int_0^t e^{-a(t-\tau)} d\tau = \frac{1}{a}(1 - e^{-at})$$

Present your solutions in the blank space provided between questions; use the back of the PRECEDING page if more space is required.

TOTAL MARKS: 50

There are 3 questions. The value for each question or part of a question is shown in parentheses next to the question number.

MARKER'S REPORT	
1	
2	
3	
TOTAL	

1. Consider the linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0$$

where

$$A = \begin{bmatrix} -5 & 2 \\ -4 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \tag{1}$$

(i) **(10 pts)** Determine the eigenvalues and eigenvectors of A. You may take the first component of each eigenvector to be 1. Use these results to determine e^{At}.

(ii) **(5 pts)** Suppose the initial condition to (1) is

$$x_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and that the input $u(t) = 1, \ 0 \leq t \leq 1$. Determine $x(1)$, the state at time 1. Leave any term of the form $e^{-\alpha}$ as is.

2(a) **(7 pts)** Consider the following (A, B) pair:

$$A = \begin{bmatrix} -\alpha & \sigma & 0 & 0 \\ 0 & -\beta & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Determine the simplest necessary and sufficient conditions for (A, B) to be controllable in terms of the parameters α, σ, and β.

2(b) **(6 pts)** Consider the linear system $\dot{x} = Ax + bu,\ x(0) = x_0$ with

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

Does there exist a control input u such that for **every nonzero** initial condition x_0, the state x can be brought to 0 in a finite time $T > 0$? Justify your answer.

3. Consider the multi-input linear system

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & -4 \end{bmatrix}; \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

The pair (A, B) is known to be controllable, but (A, b_i) is not controllable $i = 1, 2$, where b_i is the ith column of B. It is desired to find a feedback gain K such that the eigenvalues of $A - BK$ are placed at $-4, -4, -4$. Follow the steps below to find K.

(i) **(8 pts)** Using the notation of the course notes, find the matrices Q, S, and K_1 such that for $A_1 = A - BK_1$, the pair (A_1, b_1) is controllable.

(ii) **(7 pts)** Determine the transformation V which transforms (A_1, b_1) into controllable canonical form.

(iii) **(5 pts)** Using the result of (ii), determine the feedback gain k_1^T such the eigenvalues of $A_1 - b_1 k_1^T$ are at $-4, -4, -4$. You may use the fact that for any scalars a and b, the following matrices are inverses of each other

$$\begin{bmatrix} a & b & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -a & -b \end{bmatrix}$$

(iv) **(2 pts)** Combine the results of (i) and (iii) to determine the overall desired gain K.