UNIVERSITY OF TORONTO
Department of Electrical and Computer Engineering
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Solutions to Problem Set 5

1. Problem 3.1

(i)

$$\begin{aligned}(I+AB)(I-A(I+BA)^{-1}B) &= I+AB-A(I+BA)(I+BA)^{-1}B\\ &= I\end{aligned}$$

(ii) (a)

$$\begin{aligned}(A+BCD)^{-1} &= A^{-1}(I+BCDA^{-1})^{-1}\\ &= A^{-1}[I-BC(I+DA^{-1}BC)^{-1}DA^{-1}]\\ &= A^{-1}-A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}\end{aligned}$$

(b) Identify $A=P_1^{-1}$, $B=C^T$, $C=R^{-1}$, $D=C$ and applying (a) immediately gives the required results.

(c)

$$\begin{aligned}P_2C^TR^{-1} &= P_1C^TR^{-1}-P_1C^T(CP_1C^T+R)^{-1}CP_1C^TR^{-1}\\ &= P_1C^TR^{-1}-P_1C^TR^{-1}(CP_1C^TR^{-1}+I)^{-1}CP_1C^TR^{-1}\\ &= P_1C^TR^{-1}[I-(CP_1C^TR^{-1}+I)^{-1}CP_1C^TR^{-1}]\\ &= P_1C^TR^{-1}(I+CP_1C^TR^{-1})^{-1}\\ &= P_1C^T(CP_1C^T+R)^{-1}\end{aligned}$$

2. Problem 3.2

(i) For $q=0$, the Riccati difference equation is given by

$$p_{k+1}=a^2p_k-\frac{a^2p_k^2}{p_k+r}=\frac{a^2p_kr}{p_k+r}$$

Hence

$$p_{k+1}^{-1}=\frac{1}{a^2}p_k^{-1}+\frac{1}{a^2r}$$

The equation in p_k^{-1}, being linear, can be solved explicitly.

$$\begin{aligned}p_k^{-1} &= \frac{1}{a^{2k}}p_0^{-1}+\frac{1}{a^2r}\sum_{j=0}^{k-1}\frac{1}{a^{2j}}\\ &= \frac{1}{a^{2k}p_0}+\frac{1}{a^2r}\frac{1-\frac{1}{a^{2k}}}{1-\frac{1}{a^2}}\\ &= \frac{1}{a^{2k}}[\frac{1}{p_0}+\frac{a^{2k}-1}{r(a^2-1)}]\\ &= \frac{1}{a^{2k}}\frac{r(a^2-1)+p_0(a^{2k}-1)}{p_0r(a^2-1)}\end{aligned}$$

Hence

$$p_k = \frac{a^{2k} p_0 r(a^2-1)}{r(a^2-1) + p_0(a^{2k}-1)}$$

For $p_0 \neq 0$, p_k converges to 0 if $|a| < 1$, and converges to $r(a^2-1)$ if $|a| > 1$. The algebraic Riccati equation is given by

$$p = \frac{a^2 pr}{p+r}$$

This gives two roots: $p = 0$ and $p = r(a^2-1)$. For $|a| < 1$, p_k converges to the unique positive semidefinite solution of the ARE. For $|a| > 1$, p_k converges to the positive solution of the ARE, even though the system is not stabilizable.

(ii) First note that

$$a - \frac{ap_k}{p_k + r} = \frac{ar}{p_k + r}$$

Hence if $p_0 \neq 0$, then for $|a| < 1$, $\frac{ar}{p_k+r}$ converges to a, which is stable. For $|a| > 1$, $\frac{ar}{p_k+r}$ converges to $\frac{1}{a}$, which is also stable. Since we have detectability but not stabilizability in this case, we conclude that these conditions are sufficient but not necessary for stability of the time-invariant filter.

3. Problem 3.3

(i) As long as v_k is uncorrelated with x_k, the following relation for measurement update of error covariance matrix holds.

$$P_{k|k} = P_{k|k-1} - P_{k|k-1}C^T(CP_{k|k-1}C^T + HRH^T)^{-1}CP_{k|k-1}$$

(ii) To express $P_{k+1|k}$ in terms of $P_{k|k}$, we proceed as follows.

(a) Do orthogonal projection of both sides of the output equation onto y^k. This gives rise to

$$\begin{aligned} y_k - C\hat{x}_{k|k} &= H\hat{v}_{k|k} \\ &= HE(v_k\tilde{y}_{k|k-1}^T)(CP_{k|k-1}C^T + HRH^T)^{-1}\tilde{y}_{k|k-1} \\ &= HRH^T(CP_{k|k-1}C^T + HRH^T)^{-1}\tilde{y}_{k|k-1} \end{aligned}$$

Hence

$$(HRH^T)^{-1}(y_k - C\hat{x}_{k|k}) = (CP_{k|k-1}C^T + HRH^T)^{-1}\tilde{y}_{k|k-1}$$

(b) Substituting the results of (a) into the Kalman filter equations, we find

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} + G\hat{w}_{k|k} \\ &= A\hat{x}_{k|k} + GTH^T(CP_{k|k-1}C^T + HRH^T)^{-1}\tilde{y}_{k|k-1} \\ &= A\hat{x}_{k|k} + GTH^T(HRH^T)^{-1}(y_k - C\hat{x}_{k|k}) \end{aligned}$$

Define $\check{A} = A - GTH^T(HRH^T)^{-1}C$. The estimation error equation is

$$\begin{aligned} \tilde{x}_{k+1|k} &= \check{A}\tilde{x}_{k|k} + Gw_k - GTH^T(HRH^T)^{-1}Hv_k \\ &= \check{A}\tilde{x}_{k|k} + \xi_k \end{aligned}$$

(c) Now

$$E\xi_k\xi_k^T = GQG^T - GTH^T(HRH^T)^{-1}HT^TG^T$$

Define $\check{G} = G(Q - TH^T(HRH^T)^{-1}HT^T)^{\frac{1}{2}}$. Then we can write

$$\tilde{x}_{k+1|k} = \check{A}\tilde{x}_{k|k} + \check{G}\bar{\xi}_k$$

with $\bar{\xi}_k$ a white noise process with covariance equal to I. Note that ξ_k is orthogonal to Hv_k and hence also to y_k. Therefore it is orthogonal to $\tilde{x}_{k|k}$. Thus we find

$$P_{k+1|k} = \check{A} P_{k|k} \check{A}^T + \check{G}\check{G}^T$$

which is the desired equation.

4. Problem 3.4

It is easily verified that (C, A) is observable, hence detectable. Since $T = 1$, we need to check stabilizability of $\check{A}, \check{G}$.

$$\begin{aligned} \check{A} &= A - GTH^T(HRH^T)^{-1}C \\ &= A - GC = \begin{bmatrix} 1 & 1 \\ -0.5 & 0 \end{bmatrix} \end{aligned}$$

while

$$\check{G} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Hence $(\check{A}, \check{G})$ is stabilizable if and only if $\check{A}$ is stable. Now the eigenvalues of $\check{A}$ are at $0.5 \pm 0.5i$ which are stable. $(\check{A}, \check{G})$ is therefore stabilizable. Using $(ex3.1)$, the form of the algebraic Riccati equation, in this case with $\check{G} = 0$, is given by

$$P = \check{A}P\check{A}^T - \check{A}PC^T(CPC^T + HRH^T)^{-1}CP\check{A}^T$$

By inspection $P = 0$ is a solution, and by Theorem 3.3, the unique solution positive semidefinite solution in this case. The steady state Kalman filter equation is therefore given by

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + G(y_k - C\hat{x}_{k|k-1})$$

(Why is this not unexpected?)

5. Problem 3.6

$y_k = z_k + v_k$

$$\begin{aligned} \Phi_z(\omega) &= \frac{0.36(2 + 2\cos\omega)}{2.04 + 0.8\cos\omega + 2\cos 2\omega)} \\ \Phi_z(\omega) &= \frac{0.36(1 + e^{i\omega})(1 + e^{-i\omega})}{2.04 + 0.4(e^{i\omega} + e^{-i\omega}) + (e^{-i2\omega} + e^{-i2\omega})} \\ &= \frac{0.36(1 + e^{i\omega})(1 + e^{-i\omega})}{(e^{2i\omega} + 0.2e^{i\omega} + 1)(e^{-2i\omega} + 0.2e^{-i\omega} + 1)} \end{aligned}$$

If we use the realization

$$\begin{aligned} x_{k+1} &= Ax_k + gw_k \\ z_k &= [1 \quad 0]x_k = Cx_k \\ A &= \begin{bmatrix} 0 & 1 \\ -1 & -0.2 \end{bmatrix} \end{aligned}$$

Then

$$\begin{aligned} c(zI - A)^{-1} &= [1 \quad 0]\begin{bmatrix} z & -1 \\ 1 & z + 0.2 \end{bmatrix}^{-1} \\ &= \frac{[1 \quad 0]\begin{bmatrix} z + 0.2 & 1 \\ -1 & z \end{bmatrix}}{z^2 + 0.2z + 1} \\ &= \frac{[z + 0.2 \quad 1]}{z^2 + 0.2z + 1} \end{aligned}$$

$$\therefore (z+0.2)g_1 + g_2 = z+1 \quad \Rightarrow \quad \begin{array}{l} g_1 = 1 \\ g_2 = 0.8 \end{array}$$

$$y_k = Cx_k + v_k$$

Since w_k and v_k are uncorrelated, we can write the steady state filtered estimate recursion in the form

$$\hat{x}_{k+1|k+1} = A\hat{x}_{k|k} + L(y_{k+1} - CA\hat{x}_{k|k})$$

where $L = PC^T(CPC^T + HRH^T)^{-1}$ and P is the steady state solution of the algebraic Riccati equation. Solving for the steady state solution using Matlab gives (note that Matlab uses M to denote the matrix L defined above):

$$P = \begin{bmatrix} 1.1758 & 0.0855 \\ 0.0855 & 0.81915 \end{bmatrix} \qquad L = \begin{bmatrix} 0.5404 \\ 0.0393 \end{bmatrix}$$

$$\begin{aligned} \hat{x}_{k+1/k+1} &= A\hat{x}_{k/k} + \begin{bmatrix} 0.5404 \\ 0.0393 \end{bmatrix} (y_{k+1} - CA\hat{x}_{k/k}) \\ &= (A - LCA)\hat{x}_{k/k} + Ly_{k+1} \\ &= \begin{bmatrix} 0 & 0.4596 \\ -1 & -0.2393 \end{bmatrix} \hat{x}_{k/k} + \begin{bmatrix} 0.5404 \\ 0.0393 \end{bmatrix} y_{k+1} \end{aligned}$$

$$\begin{aligned} \therefore \hat{x}_{k/k} &= \left\{ I - \begin{bmatrix} 0 & 0.4596z^{-1} \\ -z^{-1} & -0.2393z^{-1} \end{bmatrix} \right\}^{-1} \begin{bmatrix} 0.5404 \\ 0.0393 \end{bmatrix} y_k \\ &= \begin{bmatrix} 1 & -0.4596z^{-1} \\ z^{-1} & 1+0.2393z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 0.5404 \\ 0.0393 \end{bmatrix} \\ &= \frac{\begin{bmatrix} 1+0.2393z^{-1} & 0.4596z^{-1} \\ -z^{-1} & 1 \end{bmatrix} \begin{bmatrix} 0.5404 \\ 0.0393 \end{bmatrix} y_k}{1 + 0.2393z^{-1} + 0.4596z^{-2}} \\ \Rightarrow \quad \hat{z}_{k/k} = \hat{x}^1_{k/k} &= \frac{0.5404 + 0.1474z^{-1}}{1 + 0.2393z^{-1} + 0.4596z^{-2}} y_k \end{aligned}$$

Alternatively we could use the realization

$$x_{k+1} = \begin{bmatrix} 0 & -1 \\ 1 & -0.2 \end{bmatrix} x_k + \begin{bmatrix} -1 \\ 0.8 \end{bmatrix} w_k$$

$$\begin{aligned} y_k &= [0 \quad 1]x_k + w_k + v_k \\ &= [0 \quad 1]x_k + \xi_k \qquad E\xi_k w_k = 0.36 \end{aligned}$$

Thus

$$TH^T = 0.36$$

$$\hat{x}_{k+1/k} = A\hat{x}_{k/k} + \begin{bmatrix} -1 \\ 0.8 \end{bmatrix} G(0.36)(CPC^T + 1.36)^{-1}(y_k - C\hat{x}_{k/k-1})$$

$$\begin{aligned} \hat{x}_{k+1/k+1} &= \hat{x}_{k+1/k} + PC(HRH^T + CPC^T)^{-1}(y_{k+1} - C\hat{x}_{k+1/k}) \\ &= (I - LC)\hat{x}_{k+1/k} + Ly_{k+1} \end{aligned}$$

But

$$\begin{aligned} \hat{x}_{k+1/k} &= A\hat{x}_{k/k} + GTH^T(HRH^T)^{-1}(y_k - C\hat{x}_{k/k}) \\ &= \check{A}\hat{x}_{k/k} + GTH^T(HRH^T)^{-1}y_k \end{aligned}$$

$$\hat{x}_{k+1/k+1} = (I - LC)\breve{A}\hat{x}_{k/k} + (I - LC)GTH^T(HRH^T)^{-1}y_k + Ly_{k+1}$$

$$H = 1, \quad T = 0.36, \quad G = \begin{bmatrix} -1 \\ 0.8 \end{bmatrix}, \quad R = 1.36$$

$$P_{k+1/k} = \breve{A}P_{k/k}\breve{A}^T + \breve{G}\breve{G}^T$$

$$P_{k+1/k} = \breve{A}(P_{k/k-1} - P_{k/k-1}C^T(CP_{k/k-1}C^T + HRH^T)^{-1}CP_{k/k-1})\breve{A} + \breve{G}\breve{G}^T$$

$$\breve{A} = \begin{bmatrix} 0 & -0.7353 \\ 1 & -0.4118 \end{bmatrix}$$

$$(I - LC)\breve{A} = \begin{bmatrix} 0.0181 & -0.7427 \\ 0.6521 & -0.2574 \end{bmatrix}$$

$$\begin{aligned}
[I - z^{-1}((I - LC)\breve{A})]^{-1} &= \begin{bmatrix} 1 - z^{-1}0.0181 & 0.7427z^{-1} \\ -0.6521z^{-1} & 1 + 0.2574z^{-1} \end{bmatrix}^{-1} \\
&= \frac{\begin{bmatrix} 1 + 0.2574z^{-1} & -0.7427z^{-1} \\ 0.6521z^{-1} & 1 - z^{-1}0.0181 \end{bmatrix}^{-1}}{1 + 0.2393z^{-1} + 0.4596z^{-2}}
\end{aligned}$$

$$\begin{aligned}
\therefore C\hat{x}_{k/k} &= C(I - [(I - LC)\breve{A}]z^{-1})^{-1}[L - z^{-1}(I - LC)GTH^T(HRH^T)^{-1}]y_k \\
&= \frac{[0.6521z^{-1} \quad 1 - 0.0181z^{-1}]\left(\begin{bmatrix} 0.0181 \\ 0.3749 \end{bmatrix} + z^{-1}\begin{bmatrix} -0.2609 \\ 0.1324 \end{bmatrix}\right)}{1 + 0.2393z^{-1} + 0.4596z^{-2}} y_k
\end{aligned}$$

$$\begin{aligned}
\hat{w}_{k/k} &= GTH^T(HRH^T)^{-1}(y_k - C\hat{x}_{k/k}) \\
&= L(y_k - C\hat{x}_{k/k})
\end{aligned}$$

Finally,

$$\hat{z}_{k/k} = C\hat{x}_{k/k} + \hat{w}_{k/k}$$

and can be found by combining all terms needed!!

6. Problem 3.7

The augmented system is given by

$$\begin{aligned}
\begin{bmatrix} x_{k+1} \\ \xi_{k+1} \end{bmatrix} &= \begin{bmatrix} A_k & 0 \\ 0 & I \end{bmatrix}\begin{bmatrix} x_k \\ \xi_k \end{bmatrix} + \begin{bmatrix} w_k \\ 0 \end{bmatrix} \\
y_k &= \begin{bmatrix} C_k & 0 \end{bmatrix}\begin{bmatrix} x_k \\ \xi_k \end{bmatrix} + v_k
\end{aligned}$$

with initial condition

$$z_j = \begin{bmatrix} x_j \\ x_j \end{bmatrix}$$

Using Kalman filtering, we obtain the following results:

$$\begin{aligned}
z_{k+1|k} &= \begin{bmatrix} A_k & 0 \\ 0 & I \end{bmatrix} z_{k|k-1} + \begin{bmatrix} A_k & 0 \\ 0 & I \end{bmatrix}\Sigma_{k|k-1}\begin{bmatrix} C_k^T \\ 0 \end{bmatrix} * \\
&\quad \left(\begin{bmatrix} C_k & 0 \end{bmatrix}\Sigma_{k|k-1}\begin{bmatrix} C_k^T \\ 0 \end{bmatrix} + R_k\right)^{-1}(y_k - \begin{bmatrix} C_k & 0 \end{bmatrix} z_{k|k-1})
\end{aligned}$$

Partition the matrix $\Sigma_{k|k-1}$ into

$$\Sigma_{k+1|k} = \begin{bmatrix} P_{k+1|k} & \hat{\Sigma}_{k+1|k} \\ \hat{\Sigma}^T_{k+1|k} & P_{j|k} \end{bmatrix}$$

with initial condition

$$\hat{z}_{j|j-1} = \begin{bmatrix} \hat{x}_{j|j-1} \\ \hat{x}_{j|j-1} \end{bmatrix}$$

which is obtainable from the normal Kalman filter equation for x_k. From the augmented system, we obtain the smoothing equation

$$\hat{x}_{j|k} = \hat{x}_{j|k-1} + \hat{\Sigma}^T_{k|k-1} C_k^T (C_k P_{k|k-1} C_k^T + R_k)^{-1} (y_k - C_k \hat{x}_{k|k-1})$$

Similarly, the covariance equation is given by

$$\begin{aligned}
\Sigma_{k+1|k} &= \begin{bmatrix} A_k & 0 \\ 0 & I \end{bmatrix} \Sigma_{k|k-1} \begin{bmatrix} A_k^T & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} A_k & 0 \\ 0 & I \end{bmatrix} \Sigma_{k|k-1} \begin{bmatrix} C_k^T \\ 0 \end{bmatrix} * \\
&\quad \left(\begin{bmatrix} C_k & 0 \end{bmatrix} \Sigma_{k|k-1} \begin{bmatrix} C_k^T \\ 0 \end{bmatrix} + R_k \right)^{-1} \begin{bmatrix} C_k & 0 \end{bmatrix} \Sigma_{k|k-1} \begin{bmatrix} A_k^T & 0 \\ 0 & I \end{bmatrix} + \begin{bmatrix} Q_k & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} A_k P_{k|k-1} A_k^T & A_k \hat{\Sigma}_{k|k-1} \\ \hat{\Sigma}^T_{k|k-1} A_k^T & P_{j|k-1} \end{bmatrix} \\
&\quad - \begin{bmatrix} A_k P_{k|k-1} C_k^T \\ \hat{\Sigma}^T_{k|k-1} C_k^T \end{bmatrix} (C_k P_{k|k-1} C_k^T + R_k)^{-1} \begin{bmatrix} C_k P_{k|k-1} A_k^T & C_k \hat{\Sigma}_{k|k-1} \end{bmatrix} \\
&\quad + \begin{bmatrix} Q_k & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

Hence

$$\hat{\Sigma}_{k+1|k} = [A_k - A_k P_{k|k-1} C_k^T (C_k P_{k|k-1} C_k^T + R_k)^{-1} C_k] \hat{\Sigma}_{k|k-1}$$

and

$$P_{j|k} = P_{j|k-1} - \hat{\Sigma}^T_{k|k-1} C_k^T (C_k P_{k|k-1} C_k^T + R_k)^{-1} C_k \hat{\Sigma}_{k|k-1}$$

This yields

$$\begin{aligned}
P_{j|j-1} - P_{j|k} &= \Sigma_{s=j}^k (P_{j|s-1} - P_{j|s}) \\
&= \Sigma_{s=j}^k \hat{\Sigma}^T_{s|s-1} C_s^T (C_s P_{s|s-1} C_s^T + R_s)^{-1} C_s \hat{\Sigma}_{s|s-1}
\end{aligned}$$

which is nondecreasing in k.