UNIVERSITY OF TORONTO
Department of Electrical and Computer Engineering
ECE1639F Fall 2008
Problem Set #4 Solutions

1. Problem 2.2

   The observable representation of the ARMA equation is given by

$$x_{k+1} = ax_k + (c+a)w_k$$

   The solution of the Lyapunov equation for the steady state covariance Σ_∞ is easily seen to be given by

$$\Sigma_\infty = \frac{(c+a)^2}{1-a^2}$$

   Now the steady state mean square value of the output is given by

$$\begin{aligned} Ey_k^2 &= Ex_k^2 + Ew_k^2 \\ &= \frac{(c+a)^2}{1-a^2} + 1 \\ &= \frac{1+2ac+c^2}{1-a^2} \end{aligned}$$

   which is the same result as that in Section 2.7.

2. Problem 2.3

   A^k can be determined as follows:

$$0 = \left\{0.4I - \begin{bmatrix} 0.4 & 0 \\ -0.6 & 0.2 \end{bmatrix}\right\} v = \begin{bmatrix} 0 & 0 \\ 0.6 & 0.2 \end{bmatrix} v$$

$$\Rightarrow v = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$

$$0 = \left\{0.2I - \begin{bmatrix} 0.4 & 0 \\ -0.6 & 0.2 \end{bmatrix}\right\} v = \begin{bmatrix} -0.2 & 0 \\ 0.6 & 0 \end{bmatrix} v$$

$$\Rightarrow v = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

   Hence $T = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$ satisfies

$$T^{-1}AT = \begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}$$

$$\begin{aligned} A &= T\begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 0.4 & 0 \\ 0 & 0.2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \end{aligned}$$

$$\begin{aligned}
\therefore A^k &= \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 0.4^k & 0 \\ 0 & 0.2^k \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0.4^k & 0 \\ -3(0.4^k - 0.2^k) & 0.2^k \end{bmatrix}
\end{aligned}$$

$$\begin{aligned}
\Sigma_k &= A^k (A^T)^k + \sum_{j=0}^{k-1} A^j G Q G^T (A^T)^j \\
&= A^k (A^T)^k + \sum_{j=0}^{k-1} \begin{bmatrix} 0.4^j \\ -3 \times 0.4^j + 4 \times 0.2^j \end{bmatrix} \begin{bmatrix} 0.4^j & -3 \times 0.4^j + 4 \times 0.2^j \end{bmatrix} \\
&= A^k (A^T)^k + \sum_{j=0}^{k-1} \begin{bmatrix} 0.16^j & -3 \times 0.16^j + 4 \times 0.08^j \\ -3 \times 0.16^j + 4 \times 0.08^j & 9 \times 0.16^j - 24 \times 0.08^j + 16 \times 0.04^j \end{bmatrix} \\
&= A^k (A^T)^k \\
&\quad + \sum_{j=0}^{k-1} \begin{bmatrix} \frac{1-0.16^k}{1-0.16} & -3 \times \frac{1-0.16^k}{1-0.16} + 4 \times \frac{1-0.08^k}{1-0.08} \\ -3 \times \frac{1-0.16^k}{1-0.16} + 4 \times \frac{1-0.08^k}{1-0.08} & 9 \times \frac{1-0.16^k}{1-0.16} - 24 \times \frac{1-0.08^k}{1-0.08} + 16 \times \frac{1-0.04^k}{1-0.04} \end{bmatrix}
\end{aligned}$$

The stationary covariance satisfies

$$\Sigma = A \Sigma A^T + G G^T$$

Let $\Sigma = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix}$, we find

$$\begin{aligned}
\begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} &= \begin{bmatrix} 0.4 & 0 \\ -0.6 & 0.2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} \begin{bmatrix} 0.4 & -0.6 \\ 0 & 0.2 \end{bmatrix} + \\
&= \begin{bmatrix} 0.4 & 0 \\ -0.6 & 0.2 \end{bmatrix} \begin{bmatrix} 0.4\sigma_1 & -0.6\sigma_1 + 0.2\sigma_2 \\ 0.4\sigma_2 & -0.6\sigma_2 + 0.2\sigma_3 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}
\end{aligned}$$

so that

$$\begin{aligned}
\sigma_1 &= 0.16\sigma_1 + 1 \\
\sigma_2 &= -0.24\sigma_1 + 0.08\sigma_2 + 1 \\
\sigma_3 &= 0.36\sigma_1 - 0.24\sigma_2 + 0.04\sigma_3 + 1
\end{aligned}$$

$$\begin{aligned}
\therefore \sigma_1 &= \frac{1}{0.84} = 1.19 \\
\sigma_2 &= 0.776 \\
\sigma_3 &= 1.294
\end{aligned}$$

It is now easily verified that $\Sigma_k \to \Sigma$, where Σ_k is the covariance function determined previously.

For the output correlation function of the resulting stationary process, we can use the results of Section 2.7. This gives, $k > 0$,

$$\begin{aligned}
R_y(k) &= [0\ 1] \begin{bmatrix} 0.4^k & 0 \\ -3(0.4^k - 0.2^k) & 0.2^k \end{bmatrix} \begin{bmatrix} 1.19 & 0.776 \\ 0.776 & 1.294 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= [-3(0.4^k - 0.2^k)\ 0.2^k] \begin{bmatrix} 0.776 \\ 1.294 \end{bmatrix} \\
&= -2.329 \times 0.4^k + 3.623 \times 0.2^k
\end{aligned}$$

Similarly, for $k < 0$,

$$\begin{aligned} R_y(k) &= C\Sigma(A^T)^{-k}C^T \\ &= [0\ 1][\Sigma(A^T)^{-k}]\begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= [CA^{-k}\Sigma C^T]^T = -2.329 \times 0.4^{-k} + 3.623 \times (0.2)^{-k} \end{aligned}$$

so that

$$R_y(k) = -2.329 \times 0.4^{|k|} + 3.623 \times 0.2^{|k|}$$

3. Problem 2.5

In general, let the stochastic difference equation be given by

$$y_k + a_1 y_{k-1} + a_2 y_{k-2} = w_k$$

where w_k is zero mean, i.i.d. sequence with variance σ^2. The roots of the polynomial $z^2 + a_1 z + a_2$, λ_1 and λ_2, are distinct and lie in $|z| < 1$.

(i) The observable representation is given by

$$\begin{aligned} x_{k+1} &= \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix} x_k + \begin{bmatrix} -a_2 \\ -a_1 \end{bmatrix} w_k \\ y_k &= \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + w_k \end{aligned}$$

Since the process y_k is stationary, x_k is also stationary. So $\Sigma = A\Sigma A^T + \begin{bmatrix} a_2^2 & a_1 a_2 \\ a_1 a_2 & a_1^2 \end{bmatrix} \sigma^2$

Let

$$\Sigma = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix}$$

Then

$$\begin{aligned} \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} &= \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix} \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} + \begin{bmatrix} a_2^2 & a_1 a_2 \\ a_1 a_2 & a_1^2 \end{bmatrix} \sigma^2 \\ &= \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix} \begin{bmatrix} -a_2\sigma_2 & \sigma_1 - a_1\sigma_2 \\ -a_2\sigma_3 & \sigma_2 - a_1\sigma_3 \end{bmatrix} + \begin{bmatrix} a_2^2 & a_1 a_2 \\ a_1 a_2 & a_1^2 \end{bmatrix} \sigma^2 \\ &= \begin{bmatrix} a_2^2\sigma_3 & -a_2\sigma_2 + a_1 a_2 \sigma_3 \\ -a_2\sigma_2 + a_1 a_2\sigma_3 & \sigma_1 - 2a_1\sigma_2 + a_1^2\sigma_3 \end{bmatrix} + \begin{bmatrix} a_2^2 & a_1 a_2 \\ a_1 a_2 & a_1^2 \end{bmatrix} \sigma^2 \end{aligned}$$

This gives the equations

$$\begin{aligned} \sigma_1 &= a_2^2(\sigma^2 + \sigma_3) \\ \sigma_2 &= -a_2\sigma_2 + a_1 a_2(\sigma^2 + \sigma_3) \\ \sigma_3 &= \sigma_1 - 2a_1\sigma_2 + a_1^2(\sigma^2 + \sigma_3) \end{aligned}$$

We can express

$$\begin{aligned} \sigma^2 + \sigma_3 &= \frac{\sigma_1}{a_2^2} \\ \sigma_2 &= \frac{a_1 a_2}{1 + a_2}(\sigma^2 + \sigma_3) = \frac{a_1}{(1 + a_2)a_2}\sigma_1 \end{aligned}$$

Adding σ^2 to both sides of the σ_3 equation gives

$$\frac{\sigma_1}{a_2^2} = \sigma_1 - 2a_1\frac{a_1}{(1+a_2)a_2}\sigma_1 + \frac{a_1^2}{a_2^2}\sigma_1 + \sigma^2$$

This gives

$$\begin{aligned}
\sigma_1(\frac{1}{a_2^2} - 1 + \frac{2a_1^2}{(1+a_2)a_2} - \frac{a_1^2}{a_2^2}) &= \sigma^2 \\
\sigma_1\frac{(1+a_2) - (1+a_2)a_2^2 + 2a_1^2a_2 - a_1^2(1+a_2)}{(1+a_2)a_2^2} &= \sigma^2
\end{aligned}$$

Solving for σ_1 yields

$$\sigma_1 = \frac{(1+a_2)a_2^2}{1+a_2-a_2^2-a_2^3+a_1^2a_2-a_1^2}\sigma^2$$

Hence

$$\sigma_2 = \frac{a_1a_2}{1+a_2-a_2^2-a_2^3+a_1^2a_2-a_1^2}\sigma^2$$

$$\sigma_3 = \frac{a_1^2+a_2^2+a_2^3-a_1^2a_2}{1+a_2-a_2^2-a_2^3+a_1^2a_2-a_1^2}\sigma^2$$

For $a_1 = 1.1$, $a_2 = 0.24$, and $\sigma^2 = 1$,

$$\sigma_1 = 0.2896 \quad \sigma_2 = 1.0603 \quad \sigma_3 = 3.9804$$

To diagonalize $A = \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix}$, we can use $P = \begin{bmatrix} -a_2 & -a_2 \\ \lambda_1 & \lambda_2 \end{bmatrix}$ where λ_1 and λ_2 are the distinct eigenvalues of A to give

$$\begin{aligned}
AP &= \begin{bmatrix} 0 & -a_2 \\ 1 & -a_1 \end{bmatrix}\begin{bmatrix} -a_2 & -a_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} \\
&= \begin{bmatrix} -a_2\lambda_1 & -a_2\lambda_2 \\ -a_1\lambda_1 - a_2 & -a_1\lambda_2 - a_2 \end{bmatrix} = \begin{bmatrix} -a_2 & -a_2 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = P\Lambda
\end{aligned}$$

Thus

$$A^k = P\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}P^{-1} = \begin{bmatrix} -a_2 & -a_2 \\ \lambda_1 & \lambda_2 \end{bmatrix}\begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix}\frac{\begin{bmatrix} \lambda_2 & a_2 \\ -\lambda_1 & -a_2 \end{bmatrix}}{(\lambda_1-\lambda_2)a_2}$$

For $k = 0$, $R_y(0) = \sigma_3 + 1 = 4.9804$. Also, $\lambda_1 = -0.3$ and $\lambda_2 = -0.8$. For $k > 0$, we can use eq. (2.54) to get (noting $Q = 1$, $H = 1$ in this case)

$$\begin{aligned}
R_y(k) &= CA^k\Sigma C^T + CA^{k-1}G \\
&= CP\Lambda^kP^{-1}\Sigma C^T + CP\Lambda^{k-1}P^{-1}G \\
&= \begin{bmatrix} -0.3 & -0.8 \end{bmatrix}\Lambda^k\begin{bmatrix} 0.8918 \\ -5.3099 \end{bmatrix} + \begin{bmatrix} \lambda_1 & \lambda_2 \end{bmatrix}\Lambda^{k-1}\begin{bmatrix} -0.6 \\ 1.6 \end{bmatrix} \\
&= (-0.3 \times 0.8918 - 0.6)\lambda_1^k + (0.8 \times 5.3099 + 1.6)\lambda_2^k \\
&= -0.8676 \times (-0.3)^k + 5.848 \times (-0.8)^k
\end{aligned}$$

(ii) We can write

$$\begin{aligned}
x_k &= \begin{bmatrix} y_k \\ y_{k-1} \end{bmatrix} \\
&\Rightarrow\ x_{k+1} = \begin{bmatrix} y_{k+1} \\ y_k \end{bmatrix} = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix}w_{k+1}
\end{aligned}$$

The Lyapunov equation is now given by

$$\Sigma = A\Sigma A^T + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \sigma^2$$

Let

$$\Sigma = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix}$$

Then

$$\begin{aligned}
\begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} &= \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} + \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -a_1\sigma_1 - a_2\sigma_2 & \sigma_1 \\ -a_1\sigma_2 - a_2\sigma_3 & \sigma_2 \end{bmatrix} + \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix} \\
&= \begin{bmatrix} a_1^2\sigma_1 + 2a_1a_2\sigma_2 + a_2^2\sigma_3 & -a_1\sigma_1 - a_2\sigma_2 \\ -a_1\sigma_1 - a_2\sigma_2 & \sigma_1 \end{bmatrix} + \begin{bmatrix} \sigma^2 & 0 \\ 0 & 0 \end{bmatrix}
\end{aligned}$$

We get $\sigma_1 = \sigma_3$, $\sigma_2 = -a_1\sigma_1 - a_2\sigma_2$

$$\begin{aligned}
\sigma_1 &= a_1^2\sigma_1 + 2a_1a_2\sigma_2 + a_2^2\sigma_3 + \sigma^2 \\
\sigma_2 &= -\frac{a_1\sigma_1}{1 + a_2}
\end{aligned}$$

$$\sigma_1 \left(1 - a_1^2 - a_2^2 + \frac{2a_1^2a_2}{1 + a_2} \right) = \sigma^2$$

$$\begin{aligned}
\sigma_1 &= \frac{(1 + a_2)\sigma^2}{1 - a_1^2 - a_2^2 + a_2 + a_1^2a_2 - a_2^3} \\
\sigma_2 &= -\frac{a_1\sigma^2}{1 - a_1^2 - a_2^2 + a_2 + a_1^2a_2 - a_2^3}
\end{aligned}$$

For $a_1 = 1.1$, $a_2 = 0.24$, $\sigma^2 = 1$,

$$\sigma_1 = 4.9804 \quad \sigma_2 = -4.4181$$

To diagonalize $A = \begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix}$, we can use $\begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix}$ to give

$$\begin{aligned}
&\begin{bmatrix} -a_1 & -a_2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -a_1\lambda_1 - a_2 & -a_1\lambda_2 - a_2 \\ \lambda_1 & \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}
\end{aligned}$$

Thus

$$A^k = \begin{bmatrix} \lambda_1 & \lambda_2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \frac{\begin{bmatrix} 1 & -\lambda_2 \\ -1 & \lambda_1 \end{bmatrix}}{\lambda_1 - \lambda_2}$$

$$y_k = [1\ 0]x_k \Rightarrow R_y(l) = [1\ 0]R_x(l) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$
\begin{aligned}
R_x(k) &= A^k\Sigma \Rightarrow R_y(k) = [1\ 0]\begin{bmatrix}\lambda_1 & \lambda_2\\ 1 & 1\end{bmatrix}\begin{bmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{bmatrix}\frac{\begin{bmatrix}1 & -\lambda_2\\ -1 & \lambda_1\end{bmatrix}}{\lambda_1-\lambda_2} \\
&\quad \times \begin{bmatrix}\sigma_1 & \sigma_2\\ \sigma_2 & \sigma_1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix} \\
&= [\lambda_1\ \lambda_2]\begin{bmatrix}\lambda_1^k & 0\\ 0 & \lambda_2^k\end{bmatrix}\frac{\begin{bmatrix}1 & -\lambda_2\\ -1 & \lambda_1\end{bmatrix}}{\lambda_1-\lambda_2}\begin{bmatrix}\sigma_1\\ \sigma_2\end{bmatrix} \\
&= \frac{1}{\lambda_1-\lambda_2}[\lambda_1^{k+1}\ \lambda_2^{k+1}]\begin{bmatrix}\sigma_1-\lambda_2\sigma_2\\ -\sigma_1+\lambda_1\sigma_2\end{bmatrix} \\
&= \frac{1}{\lambda_1-\lambda_2}[(\sigma_1-\lambda_2\sigma_2)\lambda_1^{k+1}+(\lambda_1\sigma_2-\sigma_1)\lambda_2^{k+1}]
\end{aligned}
$$

Since the roots of the char. eq. are -0.8 and -0.3, substituting we get

$$R_y(k) = -0.8676\times(-0.3)^k + 5.848\times(-0.8)^k$$

(iii) For the contour integration approach, we can write

$$y_k = \frac{1}{(1+0.8z^{-1})(1+0.3z^{-1})}w_k$$

Hence the correlation function $R_y(k) = r_k$ is given by

$$
\begin{aligned}
r_k &= \frac{1}{2\pi i}\oint \frac{z^{k-1}}{(1+0.8z^{-1})(1+0.3z^{-1})(1+0.8z)(1+0.3z)}dz \\
&= \frac{1}{2\pi i}\oint \frac{z^{k+1}}{(z+0.8)(z+0.3)(1+0.8z)(1+0.3z)}dz \\
&= \frac{(-0.8)^{k+1}}{(-0.5)(0.36)(0.76)} + \frac{(-0.3)^{k+1}}{(0.5)(0.76)(0.91)} \\
&= 5.848(-0.8)^k - 0.8676(-0.3)^k
\end{aligned}
$$

This agrees with the results computed previously.