UNIVERSITY OF TORONTO
Department of Electrical and Computer Engineering
ECE1639F Fall 2008
Problem Set #3 Solutions

1. (i) Consider the linear matrix equation

$$A\text{cov}(Y) = \text{cov}(X, Y) \tag{1}$$

Using the hint, for a solution to exist, we require

$$\mathcal{N}(\text{cov}(Y)) \subset \mathcal{N}(\text{cov}(X, Y))$$

Now $v \in \mathcal{N}(\text{cov}(Y)) \Rightarrow (Y - m_Y)^T v = 0$ a.e. This implies $\text{cov}(X, Y)v = 0$ also. Hence $\mathcal{N}(\text{cov}(Y)) \subset \mathcal{N}(\text{cov}(X, Y))$, so that a solution to (1) exists.

(ii) Suppose there are 2 solutions A_1 and A_2 to (1). Then A_1 and A_2 both satisfy

$$\begin{aligned} A_1 \text{ cov}(Y) &= \text{cov}(X, Y) \\ A_2 \text{ cov}(Y) &= \text{cov}(X, Y) \end{aligned}$$

Now all solutions are of the form

$$A_1 + \bar{A} \text{ where } \bar{A} \text{ cov}(Y) = 0$$

Then

$$\bar{A}E(Y - m_Y)(Y - m_Y)^T \bar{A}^T = 0$$

i.e.

$$\bar{A}(Y - m_Y) = 0 \quad w.p.1.$$

Thus

$$A_1(Y - m_Y) = A_2(Y - m_Y) \quad w.p.1.$$

2. For $z_k = \begin{bmatrix} x_k & \xi_k \end{bmatrix}^T$, we have

$$\begin{aligned} z_{k+1} &= \begin{bmatrix} x_{k+1} & x_k & w_k \end{bmatrix}^T \\ &= \begin{bmatrix} f_k(z_k, w_k) \\ \begin{bmatrix} I & 0 & 0 \end{bmatrix} z_k \\ w_k \end{bmatrix} \end{aligned}$$

The equation is of the form $z_{k+1} = g_k(z_k, w_k)$, with the 3 components of g_k defined by the right hand side of the equation.

3. (a) The mean value m_k satisfies the equation

$$m_{k+1} = am_k + 1, \quad m_0 = 0$$

Solving, we obtain,

$$m_k = \sum_{j=0}^{k-1} a^{k-j-1} = \sum_{i=0}^{k-1} a^i = \frac{a^k - 1}{a - 1}$$

The covariance σ_k satisfies the equation

$$\sigma_{k+1} = a^2 \sigma_k + 1, \quad \sigma_0 = 1$$

Solving, we obtain

$$\sigma_k = a^{2k} + \sum_{j=0}^{k-1} a^{2(k-j-1)} = \sum_{i=0}^{k} a^{2i} = \frac{a^{2(k+1)} - 1}{a^2 - 1}$$

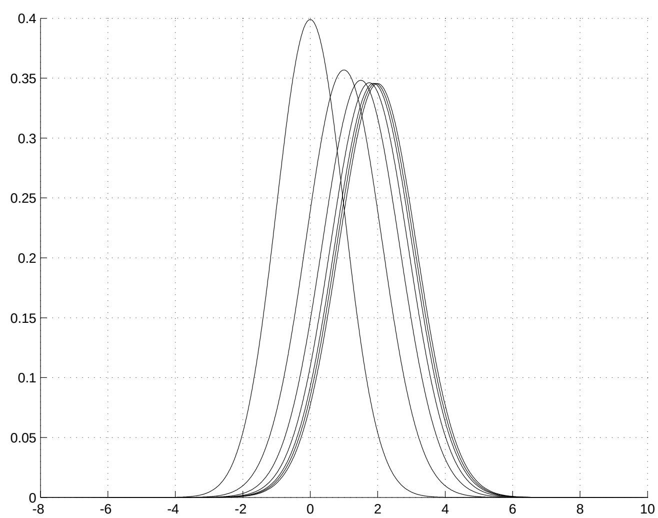
(b) For $|a| < 1$, $a^k \to 0$ as $k \to \infty$, so that

$$\lim_{k \to \infty} m_k = \frac{1}{1-a}$$

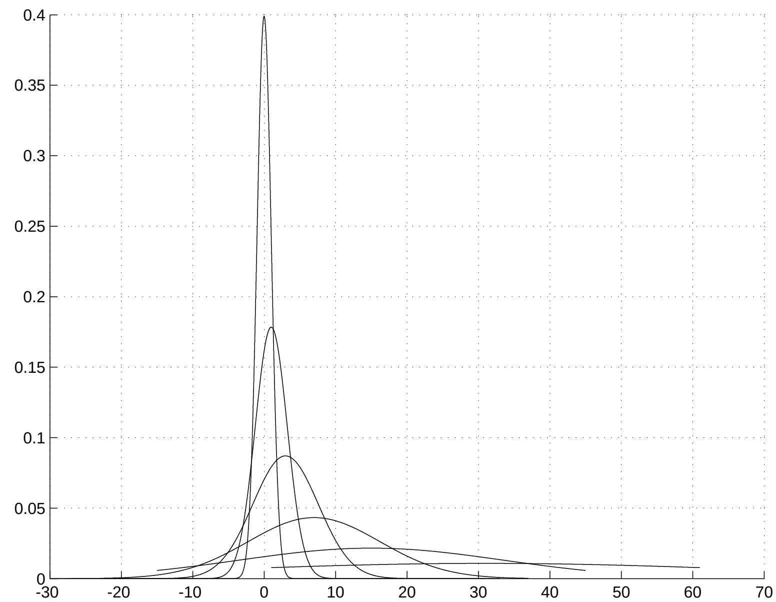
and

$$\lim_{k \to \infty} \sigma_k = \frac{1}{1-a^2}$$

For $a = \frac{1}{2}$, $m_\infty = 2$ and $\sigma_\infty = \frac{4}{3}$. A sketch of the density function of x_k for $k = 0, 1, 2, 3, 4, 5$ as well as $k = \infty$ is given in the following figure.



(c) For $a=2$, $m_k = 2^k - 1$ and $\sigma_k = \frac{4^{k+1}-1}{3}$, which grows exponentially. A sketch of the density function of x_k for $k = 0, 1, 2, 3, 4, 5$ is given in the following figure.



Note the shifting and spreading and flattening of the density function. The more spread-out and flatter the density function is, the higher the probability of getting large values. This is the probabilistic interpretation of instability.

4. Problem 2.1

(a)

$$x_{k+1} = \begin{bmatrix} 1 & 0 \\ 0.4 & 0.5 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k$$

It is easy to check that

$$\begin{aligned} \mathcal{Z}(A^k) &= z(zI - A)^{-1} \\ &= z \begin{bmatrix} z-1 & 0 \\ -0.4 & z-0.5 \end{bmatrix}^{-1} \\ &= z \begin{bmatrix} \frac{1}{z-1} & 0 \\ \frac{0.4}{(z-1)(z-0.5)} & \frac{1}{z-0.5} \end{bmatrix} \end{aligned}$$

$$\therefore \qquad \begin{aligned} A^k &= \mathcal{Z}^{-1} \begin{bmatrix} \frac{z}{z-1} & 0 \\ 0.8(\frac{z}{z-1} - \frac{z}{z-0.5}) & \frac{z}{z-0.5} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0.8(1-0.5^k) & 0.5^k \end{bmatrix} \end{aligned}$$

The solution to the covariance equation is given by

$$
\begin{aligned}
\Sigma_k &= A^k \Sigma_0 A^{T^k} + \sum_{j=0}^{k-1} A^j G G^T A^{T^j} \\
&= \begin{bmatrix} 1 & 0 \\ 0.8(1-0.5^k) & 0.5^k \end{bmatrix} \begin{bmatrix} 1 & 0.8(1-0.5^k) \\ 0 & 0.5^k \end{bmatrix} \\
&\quad + \sum_{j=0}^{k-1} \begin{bmatrix} 1 & 0 \\ 0.8(1-0.5^j) & 0.5^j \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0.8(1-0.5^j) \\ 0 & 0.5^j \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0.8(1-0.5^k) \\ 0.8(1-0.5^k) & 0.25^k + 0.64(1-0.5^k)^2 \end{bmatrix} + \sum_{j=0}^{k-1} \begin{bmatrix} 0 & 0 \\ 0 & 0.25^j \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0.8(1-0.5^k) \\ 0.8(1-0.5^k) & 0.25^k + 0.64(1-0.5^k)^2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \frac{1-0.25^k}{0.75} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0.8(1-0.5^k) \\ 0.8(1-0.5^k) & \frac{4}{3} - \frac{1}{3}(0.25^k) + 0.64(1-0.5^k)^2 \end{bmatrix}
\end{aligned}
$$

Thus

$$
\lim_{k\to\infty} \Sigma_k = \begin{bmatrix} 1 & 0.8 \\ 0.8 & 0.64 + \frac{4}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0.8 \\ 0.8 & \frac{5.92}{3} \end{bmatrix}
$$

(b) Let

$$
\Sigma = \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix}
$$

$$
\begin{aligned}
A\Sigma A^T &= \begin{bmatrix} 1 & 0 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} \sigma_1 & \sigma_2 \\ \sigma_2 & \sigma_3 \end{bmatrix} \begin{bmatrix} 1 & 0.4 \\ 0 & 0.5 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0.4 & 0.5 \end{bmatrix} \begin{bmatrix} \sigma_1 & 0.4\sigma_1 + 0.5\sigma_2 \\ \sigma_2 & 0.4\sigma_2 + 0.5\sigma_3 \end{bmatrix} \\
&= \begin{bmatrix} \sigma_1 & 0.4\sigma_1 + 0.5\sigma_2 \\ 0.4\sigma_1 + 0.5\sigma_2 & 0.16\sigma_1 + 0.4\sigma_2 + 0.25\sigma_3 \end{bmatrix}
\end{aligned}
$$

The Lyapunov equation $\Sigma = A\Sigma A^T + GG^T$ leads to the equations

$$
\begin{aligned}
\sigma_1 &= \sigma_1 \\
0.4\sigma_1 &= 0.5\sigma_2 \\
\sigma_3 &= 0.16\sigma_1 + 0.4\sigma_2 + 0.25\sigma_3 + 1 \\
0.75\sigma_3 &= 0.16\sigma_1 + 0.4\sigma_2 + 1 \\
&= 0.48\sigma_1 + 1
\end{aligned}
$$

$$
\therefore \qquad \sigma_3 = \frac{0.48}{0.75}\sigma_1 + \frac{4}{3} = 0.64\sigma_1 + \frac{4}{3}
$$

Solutions are given by

$$
\Sigma = \sigma_1 \begin{bmatrix} 1 & 0.8 \\ 0.8 & 0.64 + \frac{4}{3\sigma_1} \end{bmatrix}
$$

(c) The limit does exist, as shown in (a) and corresponds to the solution $\sigma_1 = 1$ in (b).

(d) The difference is that the A matrix here has an eigenvalue=1 and is not stable.