UNIVERSITY OF TORONTO
Department of Electrical and Computer Engineering
ECE1639F Fall 2008
Problem Set #2 Solutions

1. Problem 1.3

   Using

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{x} \quad 0 \le y \le x \le 1 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

we find the marginal densities

$$f_X x = \int_0^x \frac{1}{x} dy = 1 \quad 0 \le x \le 1$$

$$f_Y y = \int_y^1 \frac{1}{x} dx = -\log y \quad 0 \le y \le 1$$

From here, we can compute the various expectations

$$E(X) = \int_0^1 x dx = \frac{1}{2}$$

$$\begin{aligned} E(Y) &= -\int_0^1 y \log y dy \\ &= -[y^2(\frac{\log y}{2} - \frac{1}{4})]\Big|_0^1 \\ &= \frac{1}{4} \end{aligned}$$

$$E(XY) = \int_0^1 \int_0^x xy\frac{1}{x} dydx = \int_0^1 \frac{x^2}{x} dx = \frac{1}{6}$$

$$E(Y^2) = -\int_0^1 y^2 \log y dy = -[y^3(\frac{\log y}{3} - \frac{1}{9})]\Big|_0^1 = \frac{1}{9}$$

Hence

$$cov(X,Y) = E(XY) - E(X)E(Y) = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$$

$$cov(Y) = E(Y^2) - [E(Y)]^2 = \frac{1}{9} - \frac{1}{16} = \frac{7}{144}$$

The linear least squares estimate is then given by

$$\begin{aligned} \hat{X} &= \frac{1}{2} + \frac{\frac{1}{24}}{\frac{7}{144}}(Y - \frac{1}{4}) \\ &= \frac{1}{2} + \frac{6}{7}(Y - \frac{1}{4}) \\ &= \frac{6}{7}Y + \frac{2}{7} \end{aligned}$$

2. Problem 1.6

(a)

$$cov(Y) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

which is a singular matrix. This gives the relation

$$2Y_1 = Y_2 \ w.p.1$$

Thus

$$EXY_2 = 2EXY_1 = 5$$

is inconsistent with the value of $EXY_1 = 1$. The problem is not meaningful.

(b)

$$E(YY^T) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$E(YY^T)$ is singular with its nullspace spanned by the vector $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$. This implies $Y_2 = 2Y_1$ with probability 1.

(i) The equation for $\alpha = [\alpha_1 \ \ \alpha_2]^T$ is, in column vector form

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Clearly any α satisfying the equation

$$\alpha_1 + 2\alpha_2 = 1$$

is a solution. Hence we can express the general solution as

$$\alpha = \begin{bmatrix} 1 - 2\beta \\ \beta \end{bmatrix}$$

(ii) The l.l.s. estimate is given by

$$\hat{X} = \alpha^T Y = (1 - 2\beta)Y_1 + \beta Y_2$$

Noting that $Y_2 = 2Y_1$ with probability 1, we have that

$$\hat{X} = Y_1 \quad w.p.1$$

Hence $\hat{X}$ does not depend of the specific value of the free parameter β.

(iii)

$$\hat{X}_1 = \frac{E(XY_1)}{E(Y_1^2)} Y_1 = Y_1$$

The innovation $\tilde{Y}_2$ is given by

$$\tilde{Y}_2 = Y_2 - \frac{E(Y_2 Y_1)}{E(Y_1^2)} Y_1 = Y_2 - 2Y_1 = 0 \ \ w.p.1$$

Thus

$$\hat{X}_2 = \hat{X}_1$$

The reason that the innovation is zero w.p.1 is due to the fact that Y_2 is in fact a linear function of Y_1, w.p.1. Hence Y_2 does not contain any information which is not already in Y_1.

3. Problem 1.8

$$Y_i = X + V_i$$

Let

$$\hat{X} = \sum_{i=1}^{n} \alpha_i Y_i$$

Then

$$E(X - \sum_{i=1}^{n} \alpha_i Y_i) Y_j = 0$$

$$\begin{aligned} \sum_{i=1}^{n} \alpha_i E(Y_j Y_i) &= E(XY_j) \\ &= E(X^2) = {\sigma_x}^2 \end{aligned}$$

But

$$E(Y_j Y_k) = E(X + V_j)(X + V_k) = {\sigma_x}^2 + {\sigma_v}^2 \delta_{jk}$$

This leads to the normal equation

$$\begin{bmatrix} {\sigma_x}^2 + {\sigma_v}^2 & {\sigma_x}^2 & \cdots & {\sigma_x}^2 \\ {\sigma_x}^2 & {\sigma_x}^2 + {\sigma_v}^2 & \cdots & {\sigma_x}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {\sigma_x}^2 & \cdots & \cdots & {\sigma_x}^2 + {\sigma_v}^2 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = \begin{bmatrix} {\sigma_x}^2 \\ {\sigma_x}^2 \\ \vdots \\ {\sigma_x}^2 \end{bmatrix}$$

It is easily seen that $\alpha_k = \alpha$, a constant, is the unique solution to the linear system of equations. Substituting into the normal equation, we find

$$\alpha = \frac{{\sigma_x}^2}{n{\sigma_x}^2 + {\sigma_v}^2}$$

Hence, the linear least squares estimate is given by

$$\begin{aligned} \hat{X} &= \frac{{\sigma_x}^2}{n{\sigma_x}^2 + {\sigma_v}^2} \sum_{i=1}^{n} Y_i \\ &= \frac{1}{n} \frac{1}{1 + \frac{{\sigma_v}^2}{n{\sigma_x}^2}} \sum_{i=1}^{n} Y_i \\ &= \frac{1}{1 + \frac{{\sigma_v}^2}{n{\sigma_x}^2}} x + \frac{1}{n} \frac{1}{1 + \frac{{\sigma_v}^2}{n{\sigma_x}^2}} \sum_{i=1}^{n} V_i \end{aligned}$$

As $n \to \infty$, we obtain

$$\hat{X} \approx X + \frac{1}{n} \sum_{i=1}^{n} V_i$$

By the law of large numbers, $\hat{X} \to X$ as $n \to \infty$.

4. Problem 1.9

(a) Using the definition of $J(\theta)$, we get

$$
\begin{aligned}
J(\theta) - J(\hat{\theta}) &= (Y - \Phi\theta)^T Q (Y - \Phi\theta) - (Y - \Phi\hat{\theta})^T Q (Y - \Phi\hat{\theta}) \\
&= -2\theta^T \Phi^T Q Y + \theta^T \Phi^T Q \Phi \theta + 2\hat{\theta}^T \Phi^T Q Y - \hat{\theta}^T \Phi^T Q \Phi \hat{\theta} \\
&= 2(\hat{\theta} - \theta)(\Phi^T Q Y - \Phi^T Q \Phi \hat{\theta}) + (\theta - \hat{\theta})^T \Phi^T Q \Phi (\theta - \hat{\theta})
\end{aligned}
$$

We see that if $\hat{\theta}$ satisfies the normal equation

$$
\Phi^T Q \Phi \hat{\theta} = \Phi^T Q Y
$$

then

$$
J(\theta) - J(\hat{\theta}) = (\theta - \hat{\theta})^T \Phi^T Q \Phi (\theta - \hat{\theta}) \geq 0
$$

This shows that the solution of the normal equation gives the least squares parameter estimate.

(b) We show that the least squares estimate must satisfy the normal equation. We proceed using proof by contradiction. Assume that $\hat{\theta}$ is the least squares parameter estimate and that

$$
\Phi^T Q Y - \Phi^T Q \Phi \hat{\theta} \neq 0
$$

Choose θ such that

$$
\hat{\theta} - \theta = -\alpha(\Phi^T Q Y - \Phi^T Q \Phi \hat{\theta})
$$

Using (ex9.1), we see that this leads to

$$
J(\theta) - J(\hat{\theta}) = -\frac{2}{\alpha}(\hat{\theta} - \theta)^T (\hat{\theta} - \theta) + (\theta - \hat{\theta})^T \Phi^T Q \Phi (\theta - \hat{\theta})
$$

If we choose α sufficiently small, we will obtain

$$
J(\theta) - J(\hat{\theta}) < 0
$$

contradicting the optimality of $\hat{\theta}$. From these results, we see that $\hat{\theta}$ is the least squares parameter estimate if and only if it satisfies the normal equation.