UNIVERSITY OF TORONTO
Department of Electrical and Computer Engineering
ECE1639F Fall 2008
Solutions to Probability Review Problems

1. (a) Let $B_1, \cdots, B_n$ be a partition of Ω, i.e.

$$\begin{aligned} B_i \cap B_j &= \phi \\ \cup B_j &= \Omega \\ P(A) &= P(A \cap \cup B_j) \\ &= P(\cup(A \cap B_j)) = \Sigma P(A \cap B_j) \\ &= \Sigma_j P(A|B_j)P(B_j) \end{aligned}$$

(b) Let A be the event that the final ball is blue and B the event that the first ball is blue. Then

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c)$$

$$P(A|B) = \frac{5}{8}, \;\; P(B) = \frac{3}{5}$$

$$P(A|B^c) = \frac{4}{8}, \;\; P(B^c) = \frac{2}{5}$$

Hence

$$P(A) = \frac{5}{8}\frac{3}{5} + \frac{1}{2}\frac{2}{5} = \frac{3}{8} + \frac{1}{5} = \frac{23}{40}$$

2. First note that the conditional distribution function can also be written as

$$f_{X|Y}(x|y) = P(X = x, Y = y|Y = y)$$

(a)

$$\begin{aligned} E[Xg(Y)|Y = y] &= \sum_{x,u}[xg(u)P(X = x, Y = u|Y = y)] \\ &= \sum_{x,u} xg(u)\frac{P(X = x, Y = u, Y = y)}{P(Y = y)} \\ &= \sum_{x} g(y)x\frac{P(X = x, Y = y)}{P(Y = y)} \\ &= g(y)\sum_{x} xf_{X|Y}(x|y) \\ &= g(y)E(X|Y = y) \end{aligned}$$

Hence we get $E(Xg(Y)|Y) = g(Y)E(X|Y)$.

(b)

$$\begin{aligned}
E[E(X|Y,Z)|Y=y] &= \sum_z [\sum_x xP(X=x|Y=y,Z=z)P(Y=y,Z=z|Y=y)] \\
&= \sum_{z,x} x \frac{P(X=x,Y=y,Z=z)}{P(Y=y,Z=z)} \frac{P(Y=y,Z=z)}{P(Y=y)} \\
&= \sum_x x \frac{P(X=x,Y=y)}{P(Y=y)} \\
&= E(X|Y=y)
\end{aligned}$$

Hence $E[E(X|Y,Z)|Y] = E(X|Y)$.

3. We can determine $E(K|N)$ by direct computation, as follows:

$$\begin{aligned}
E(K|N=n) &= \sum_{k=0}^{n} k f_{K|N}(k|n) \\
&= \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} \\
&= \sum_{k=0}^{n} k \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} \frac{n!}{(k-1)!(n-k)!} p^k (1-p)^{n-k} \\
&= np \sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!((n-1)-(k-1))!} p^{k-1} (1-p)^{(n-1)-(k-1)} \\
&= np \sum_{m=0}^{n-1} \frac{(n-1)!}{m!((n-1)-m)!} p^m (1-p)^{(n-1)-m} \\
&= np
\end{aligned}$$

Alternatively, we observe that for all x,

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

Differentiating both sides with respect to x and multiplying the result throughout by x gives

$$\sum_{k=0}^{n} k \binom{n}{k} x^k = nx(1+x)^{n-1}$$

Set $q = 1-p$ and $x = \frac{p}{q}$. Observe that

$$\begin{aligned}
\sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} &= q^n \sum_{k=0}^{n} k \binom{n}{k} x^k \\
&= q^n n \frac{p}{q} (1 + \frac{p}{q})^{n-1} \\
&= np
\end{aligned}$$

which is the same result as before. Hence $E(K|N) = pN$. The interpretation of this result is that given there are N eggs and the probability of each egg hatching is p, we expect to get pN chicks. Furthermore, $E(K) = E(E(K|N)) = pE(N) = p\lambda$.

To get $E(N|K)$, we need to find $f_{N|K}(n|k)$. Since

$$f_{N|K}(n|k) = \frac{P(K=k, N=n)}{P(K=k)} = \frac{f_{K|N}(k|n)f_N(n)}{P(K=k)}$$

we need to find the probability mass function $f_K(k) = P(K = k)$. Now,

$$\begin{aligned}
f_K(k) &= \sum_n P(K=k, N=n) = \sum_n f_{K|N}(k|n)f_N(n) \\
&= \sum_{n\geq k} \binom{n}{k} p^k(1-p)^{n-k}\frac{\lambda^n}{n!}e^{-\lambda} \\
&= e^{-\lambda}\sum_{n=k}^{\infty}\frac{1}{(n-k)!k!}p^k(1-p)^{n-k}\lambda^n \\
&= e^{-\lambda}\frac{p^k}{k!}\sum_{j=0}^{\infty}\frac{1}{j!}(1-p)^j\lambda^{j+k} \quad \text{(by letting } n-k=j) \\
&= e^{-\lambda}\frac{p^k\lambda^k}{k!}\sum_{j=0}^{\infty}\frac{1}{j!}[(1-p)\lambda]^j \\
&= e^{-\lambda}\frac{p^k\lambda^k}{k!}e^{q\lambda}
\end{aligned}$$

Hence

$$\begin{aligned}
f_{N|K}(n|k) &= \frac{\binom{n}{k} p^k(1-p)^{n-k}\frac{\lambda^n}{n!}e^{-\lambda}}{e^{-\lambda}\frac{p^k\lambda^k}{k!}e^{q\lambda}} \\
&= \frac{q^{n-k}\lambda^{n-k}e^{-q\lambda}}{(n-k)!}
\end{aligned}$$

Finally,

$$\begin{aligned}
E(N|K=k) &= \sum_{n=k}^{\infty} n\frac{q^{n-k}\lambda^{n-k}e^{-q\lambda}}{(n-k)!} \\
&= \sum_{m=0}^{\infty}(m+k)\frac{(q\lambda)^m}{m!}e^{-q\lambda} \\
&= k + q\lambda \quad \text{(recall that the mean of a Poisson distribution with parameter } q\lambda \text{ is } q\lambda)
\end{aligned}$$

so that $E(N|K) = K + q\lambda$.

4. The transformation T taking (X_1, X_2) to (Y_1, Y_2) is given by

$$(Y_1, Y_2) = T(X_1, X_2) = (X_1 + X_2, \frac{X_1}{X_2})$$

Now, $X_1 = Y_2 X_2$ so that $Y_1 = X_2 Y_2 + X_2$. Hence $X_2 = \frac{Y_1}{1+Y_2}$ and $X_1 = \frac{Y_1 Y_2}{1+Y_2}$. Hence, the inverse of T is given by

$$(X_1, X_2) = T^{-1}(Y_1, Y_2) = (\frac{Y_1 Y_2}{1+Y_2}, \frac{Y_1}{1+Y_2})$$

The Jacobian matrix is given by

$$\frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} = \begin{bmatrix} \frac{y_2}{1+y_2} & y_1 \frac{[(1+y_2)-y_2]}{(1+y_2)^2} \\ \frac{1}{1+y_2} & -\frac{y_1}{(1+y_2)^2} \end{bmatrix}$$

Using the transformation of density formula, we get

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(x_1(y_1, y_2), x_2(y_1, y_2)) |\det \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)}| \\ &= f_{X_1,X_2}(\frac{y_1 y_2}{1+y2}, \frac{y_1}{1+y_2}) \frac{|y_1|}{(1+y_2)^2} \end{aligned}$$

Since X_1 and X_2 are nonnegative, so are Y_1 and Y_2. By the independence of X_1 and X_2, we get

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= \lambda^2 \exp\{-\lambda(\frac{y_1 y_2}{1+y2} + \frac{y_1}{1+y_2})\} \frac{y_1}{(1+y_2)^2} \\ &= \lambda^2 y_1 e^{-\lambda y_1} \frac{1}{(1+y_2)^2} = g(y_1) h(y_2) \end{aligned}$$

Since $f_{Y_1,Y_2}(y_1, y_2)$ factors into a product of a function of y_1 and a function of y_2, Y_1 and Y_2 are independent. The marginal densitites are given by

$$\begin{aligned} f_{Y_1}(y_1) &= \int_0^\infty \lambda^2 y_1 e^{-\lambda y_1} \frac{1}{(1+y_2)^2} dy_2 \\ &= \lambda^2 y_1 e^{-\lambda y_1}, \quad y_1 \geq 0 \end{aligned}$$

$$\begin{aligned} f_{Y_2}(y_2) &= \int_0^\infty \lambda^2 y_1 e^{-\lambda y_1} \frac{1}{(1+y_2)^2} dy_1 \\ &= \frac{1}{(1+y_2)^2}, \quad y_2 \geq 0 \end{aligned}$$