UNIVERSITY OF TORONTO
Department of Electrical & Computer Engineering
ECE1639F Fall 2008
Problem Set 3

Due: October 20, 2008

1. The l.l.s. estimator given by (1.6.1) assumes that $cov(Y)$ is invertible. This problem shows that even if $cov(Y)$ is not invertible, the normal equation is always solvable, and the l.l.s. estimate is unique.

   (i) Show that the equation
   $$A\ cov(Y) = cov(X, Y) \tag{1}$$
   always has a solution for A, even though in general $cov(Y)$ may be singular. When $cov(Y)$ is singular, (1) will have infinitely many solutions.
   (Hint: The equation $AM = Q$ has a solution if and only if $\mathcal{N}(M) \subset \mathcal{N}(Q)$, where $\mathcal{N}(M)$ denotes the nullspace of M.)

   (ii) In the proof of Theorem 1.2.1, the linear least squares estimate $\hat{X}$ is shown to be essentially unique in the sense that if there is another linear estimator $\hat{\hat{X}}$ such that
   $$E\|(X - \hat{X})\|^2 = E\|X - \hat{\hat{X}}\|^2$$
   then $\hat{X} = \hat{\hat{X}}$ with probability 1. (w.p.1.).
   On the other hand, we also know that if $cov(Y)$ is singular, then there will be multiple solutions to equation (1). This part shows that the particular choice of solution to (1) is immaterial:
   Show that any two solutions of (1), A_1 and A_2, satisfy
   $$A_1(Y - m_Y) = A_2(Y - m_Y) \quad w.p.1$$
   so that they give the same linear least squares estimate $m_X + A_1(Y - m_Y)$.

2. The state space model is quite general in that other seemingly different models can be reformulated into a state model. This problem illustrates the use of state augmentation for this purpose.

   Consider the equation
   $$x_{k+1} = f_k(x_k, x_{k-1}, w_{k-1}, w_k)$$
   Let $\xi_k = \begin{bmatrix} x_{k-1} & w_{k-1} \end{bmatrix}^T$. Define $z_k = \begin{bmatrix} x_k & \xi_k \end{bmatrix}^T$. Write down the equation for z_k and show that it is of the form
   $$z_{k+1} = g_k(z_k, w_k)$$
   Identify the function g.

3. A simple linear stochastic system is described by

$$x_{k+1} = ax_k + u_k + w_k$$

where $u_k = 1$ all k, x_0 is a $\mathcal{N}(0,1)$ random variable, and w_k is a zero mean Gaussian orthogonal sequence, with w_k a $\mathcal{N}(0,1)$ random variable for all k. As a result, x_k is also Gaussian.

(a) Determine m_k, the mean value of x_k, and σ_k, the covariance of x_k as a function of the paramater a.

(b) For $|a| < 1$, determine the limiting values of m_k and σ_k as k tends to ∞. For $a = \frac{1}{2}$, sketch qualitatively the density function of x_k, for $k = 0, 1, 2, 3, 4, 5$ as well as for $k = \infty$.

(c) For $a = 2$, sketch qualitatively the density function of x_k, for $k = 0, 1, 2, 3, 4, 5$. Comment on what happens as k tends to ∞.

4. Problem 2.1