UNIVERSITY OF TORONTO
Department of Electrical & Computer Engineering
ECE1639F – Fall 2008

Probability Review Problems

This problem set attempts to help you review some standard probabilistic calculations. Feel free to consult textbooks on probability if you can't solve the problems directly.

1. One of the very useful results on conditional probabilities is the following. Let $B_1, B_2, \cdots, B_n$ be a partition of the sample space Ω, i.e., $\cup_{j=1}^{n} B_j = \Omega$ and $B_i \cap B_j = \phi,\ i \neq j$.

   (a) Prove that

   $$P(A) = \sum_{j=1}^{n} P(A|B_j)P(B_j)$$

   or look up the proof in a probability text. This is sometimes referred to as the theorem of total probabilities.

   (b) As a typical application, Use the result of (a) to solve the following problem. There are 2 boxes. Box I contains 2 white balls and 3 blue balls. Box II contains 3 white balls and 4 blue balls. A ball is drawn at random from box I and placed into box II. Then a ball is drawn at random from box II. What is the probability that this second ball is blue?

2. Conditioning on discrete random variables is usually easier to study, especially through the use of the probability mass function for discrete random variables:

   $$f_X(x) = P(X = x)$$

   The usual definitions of conditioning apply to discrete random variables. The conditional probability mass function is given by

   $$f_{Y|X}(y|x) = \frac{f_{X,Y}(x, y)}{f_X(x)}$$

   The conditional expectation is given by

   $$E(Y|X = a) = \sum_y y f_{Y|X}(y|a) \ \text{ with } f_X(a) > 0$$

   From this we can define $E(Y|X)$.

   For X, Y, and Z discrete random variables, prove that

   (a)

   $$E(Yg(X)|X) = g(X)E(Y|X)$$

   (b)

   $$E(E(Y|X, Z)|X) = E(Y|X)$$

3. A discrete random variable N has a Poisson distribution with parameter $\lambda > 0$ if it has a probability mass function given by

$$f_N(n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

This distribution is often used to model the number of occurrences of a random event.

Consider the following situation. A hen lays N eggs with N a random variable having a Poisson distribution with parameter λ. Each egg hatches with probability p independent of other eggs. Let K denote the number of chicks hatched. We then have the following

$$f_N(n) = \frac{\lambda^n}{n!} e^{-\lambda}$$

and

$$f_{K|N}(k|n) = \left(\begin{array}{c} n \\ k \end{array} \right) p^k (1-p)^{n-k}$$

The last conditional distribution is the binomial distribution of having k successes (k chicks hatched) in n trials (given the number of eggs equals n), $\left(\begin{array}{c} n \\ k \end{array} \right)$ being the binomial coefficient $\frac{n!}{k!(n-k)!}$. Find $E(K|N)$ and $E(N|K)$.

4. This problem reviews the procedure for determining the density function of a function of a random variable. Let X be a random variable with density function $f_X(x)$. Let $Y = g(X)$ be another random variable which is a function of X. Assume that g is a one-to-one function so that its inverse h exists, i.e. $X = h(Y)$. Assume also that h is continuously differentiable. Then the density function of Y is given by

$$f_Y(y) = f_X(h(y))|h'(y)|$$

where $|a|$ is the absolute value of a.

(a) Apply this formula to find $f_Y(y)$ in the case where X is a Gaussian random variable with mean 0 and variance 1 and $Y = X^3$.

(b) We extend the results of (a) to functions of 2 random variables. Let X_1 and X_2 be 2 random variables with joint density function $f_{X_1,X_2}(x_1, x_2)$. Let T be a one-one mapping on the plane having an inverse Q, i.e.

$$(y_1, y_2) = T(x_1, x_2)$$

can be inverted to give

$$(x_1, x_2) = Q(y_1, y_2)$$

By abuse of notation, we often write the components of Q as

$$\begin{aligned} x_1 &= x_1(y_1, y_2) \\ x_2 &= x_2(y_1, y_2) \end{aligned}$$

Assume that the partial derivatives $\frac{\partial x_i}{\partial y_j}$, $i, j = 1, 2$ exist and are continuous. Define the Jacobian of the inverse transformation Q to be the determinant

$$J(y_1, y_2) = \det \begin{bmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_1}{\partial y_2} \\ \frac{\partial x_2}{\partial y_1} & \frac{\partial x_2}{\partial y_2} \end{bmatrix}$$

Then the joint density of Y_1, Y_2 is given by

$$\begin{aligned} f_{Y_1,Y_2}(y_1, y_2) &= f_{X_1,X_2}(x_1(y_1, y_2), x_2(y_1, y_2))|J(y_1, y_2)|, \quad \text{if } (y_1, y_2) \text{ is in the range of } T \\ &= 0 \ \ \text{otherwise} \end{aligned}$$

Apply this formula to the following:
Let X_1, X_2 be independent exponential random variables with parameter λ, i.e., $f_{X_i}(x_i) = \lambda e^{-\lambda x_i}, x_i \geq 0,\ i = 1, 2$. Let $Y_1 = X_1 + X_2$, $Y_2 = \frac{X_1}{X_2}$. Find the joint density of Y_1, Y_2. Are Y_1 and Y_2 independent? Find also the marginal densities of Y_1 and Y_2.