UNIVERSITY OF TORONTO
Department of Electrical and Computer Engineering
ECE 1639S - Analysis and Control of Stochastic Systems I
Midterm Examination
Spring 2008

You may only use the course notes, lecture notes you took, your study notes, problem sets and solutions in preparing your answers. The problems are of equal weight.

1. Two random variables X and Y have a joint density function given by

$$\begin{aligned} f_{XY}(x,y) &= (\alpha x + \beta y)e^{-(x+y)}, \qquad x \geq 0, \quad y \geq 0 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

   where $\alpha > 0$, $\beta > 0$, $\alpha + \beta = 1$. Let $Z = X + Y$. Find the conditional expectation $E(X|Z)$.

2. Sometimes it is convenient to represent input-output models with a state space model which has no noise term in the output equation, i.e., to have $y_k = Cx_k$. You saw this in Problem 3.6. For an nth-order ARMA model, the corresponding state space model with no observation noise will in general have to be of dimension $n+1$. However, it is always possible to represent an nth-order AR model using a state space model of dimension n with no observation noise.

   (a) Show that the AR model

$$y_k + a_1 y_{k-1} + a_2 y_{k-2} = w_k$$

   can be represented by the state space model

$$x_{k+1} = \begin{bmatrix} -a_1 & 1 \\ -a_2 & 0 \end{bmatrix} x_k + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_{k+1} \tag{2.1}$$

$$y_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k \tag{2.2}$$

   (b) Let the **stationary** AR process y_k be described by

$$y_k - \alpha^2 y_{k-2} = w_k \tag{2.3}$$

   where $0 < |\alpha| < 1$, and w_k is zero mean white noise with variance 1. Represent (2.3) using the representation (2.1) and (2.2). Determine the steady state covariance of the x_k process from the resulting state space representation and hence determine Ey_k^2.

3. Consider the MA process

$$y_k = w_k + cw_{k-1}, \quad 0 < |c| \neq 1$$

   It can be represented by the state space model

$$\begin{aligned} x_{k+1} &= cw_k \\ y_k &= x_k + w_k \end{aligned}$$

   where w_k is zero mean white noise with variance r, and the initial state $x_0 (= cw_{-1})$ has zero mean, variance $c^2 r$, and is uncorrelated with $w_k, k \geq 0$.

   (a) Determine the equations satisfied by the Kalman one-step ahead predictor $\hat{x}_{k|k-1}$ and the associated error covariance $p_{k|k-1}$. Determine the one-step ahead predictor for y, i.e., find $\hat{y}_{k|k-1}$.

(b) Set for convenience $p_k = p_{k|k-1}$. Solve for p_k explicitly. Determine the limiting values $\lim_{k\to\infty} p_k$ for $|c| < 1$ and $|c| > 1$.

(c) Write down the algebraic Riccati equation for this system and determine, for $|c| < 1$, all its positive semidefinite solutions. Next determine, for $|c| > 1$, all its positive semidefinite solutions. In these 2 cases, which ones, if any, correspond to the limiting values determined in part (b)?

(d) Discuss the stability of the steady state Kalman filters resulting from the different cases examined in part (c) and whether detectability and stabilizability conditions are satisfied in each case.

4. Consider the linear stochastic system

$$x_{k+1} = Ax_k + Gw_k = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x_k + \begin{bmatrix} 0 \\ 1 \end{bmatrix} w_k$$
$$y_k = Cx_k + v_k = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k$$

where w_k is zero mean white noise with variance 1, v_k is zero mean white noise with variance 1, both uncorrelated with x_0, with $Ew_k v_j = -1\delta_{kj}$.

(a) Determine $\breve{A}$ and $\breve{G}$, and determine whether the conditions of stabilizability of $(\breve{A}, \breve{G})$ and detectability of (C, A) are satisfied.

(b) Write the associated algebraic Riccati equation in the form

$$P = \breve{A}P\breve{A}^T - \breve{A}PC^T(CPC^T + HRH^T)^{-1}CP\breve{A}^T + \breve{G}\breve{G}^T$$

Solve for P and determine whether or not there exists a unique positive semidefinite solution. For each solution P, determine whether or not the corresponding $A - KC$ is stable.

(c) In terms of Theorem 3.3, what does the results of this problem illustrate with respect to the existence and uniqueness of positive semidefinite solutions to the ARE?